IJST EXPLORE International Journal of Science and Technology
CENTRE OF PROFESSIONAL RESEARCH PUBLICATIONS
ISSN 2049-7318
International Journal of Science and Technology Volume 5 No. 11, November, 2016

An Exact Solution of the Differential Equation For flow-Loaded Ropes

Mathias Paschen
Chair of Ocean Engineering, University of Rostock,
Albert-Einstein-Strasse 2, D-18059 Rostock, Germany

ABSTRACT

Flow loaded threads, ropes, cables, etc. are usually computed as discretised tension systems. The application of methods of continuum mechanics often fails due to the nonlinearity of the differential equations describing the system. The catenary curve is the only exact solution known until now. This mathematical model requires significant physical assumptions: The thread is ideal. That means it is flexible but non-elastic; its mass is continuously distributed along its length. Flow induced loads do not occur, or they are negligible. In a majority of ocean engineering applications the effects of hydrodynamic loads on threads, cables, etc. are not negligible. Therefore, the hydrodynamic loads have to be considered. Considering a rope whose weight is compensated by its hydrostatic buoyancy, a further solution of the differential equations can be specified. This exact solution can be used for the validation of results from various numerical models.

Keywords: *Ideal Thread, flow-Loaded Cable, Differential Equation, Explicit Solution, Validation of Numerical Models*

NOMENCLATURE

F_i, F	vector, absolute value of tension $[N]$
R_i, R	vector, absolute value of hydrodynamic load $[N]$
Rn_d	Reynolds number formed with the diameter of a circular cylinder
c_{d-90}	hydrodynamic drag coefficient of a circular cylinder in the case of cross-flow
d	diameter of the thread $[m]$
$e_{t,i}$	vector of the thread tangent
i	represent the axis x, y, z
q_i, q	vector, absolute value of weight of rope per meter in water $[N \cdot m^{-1}]$
r_i	position vector $[m]$
s	parameter, coordinate along the thread $[m]$
t	time $[s]$
v_i, v	vector, absolute value of current velocity $[m \cdot s^{-1}]$
$v_{N,i}, v_N$	vector, absolute value of normal vector $[m \cdot s^{-1}]$
ρ	density of the fluid $[kg \cdot m^{-3}]$

1. INTRODUCTION

The calculation of ropes, hawsers, chains and cables is often based on the assumption of an ideal thread. The ideal thread is described as a spatial continuum with an infinite number of degrees of freedom. Because its diameter is negligibly small compared to its centre line, it is admissible to characterize its movement by the movement of its centre line. This represents a three-dimensional curve. It is defined at any time by the function $r_i = r_i(s,t)$ if r_i is the position vector, s a parameter and t the time.

The thread is ideal. In this context, the term *ideal* refers to the kinematic and dynamic properties of the thread: it is considered to be inextensible in longitudinal direction and bending takes place without any resistance. Because of this property, the tangent of the ideal thread always has the same orientation as the tension.

Many scientists have already dealt with the theory of the ideal thread. Hamel [2] pointed out that Lagrange was probably the first one.

In recent decades, the theory of the ideal thread was often the scientifically based method for calculating flexible systems for subsea and marine applications. Stengel [3], for instance, created a mathematical model for the calculation of deflection and tension of heavy, inelastic and flow-loaded ropes and cables in steady current. Unfortunately, the derived differential equations are non-linear and it is in general not possible to find an explicit solution. Regardless of the fact that numerical integration is required, this gave the computer-aided development of fishing gears a considerable impulse at that time. However, for many other applications like the analysis of mooring lines [1] it is a good approximation to neglect the effect of hydrodynamic loads. This simplifies the analysis and leads to the well-known catenary.

In many marine applications, steel cables are increasingly substituted by modern high-strength textile ropes. The density of modern manmade fiber materials is generally much less than that of steel. Aramide fibres have almost the same density as seawater. That means weight and buoyancy of this material are almost in balance. Under this condition, the tensile force and the hydrodynamic loads are the only components of the mathematical model by Stengel.

The aim of this paper is to show that it is possible to find an explicit solution for the differential equations if the tangential component of the hydrodynamic load is negligibly small compared to the normal component.

2. MATERIAL AND METHODS

2.1. Original model

We consider the equilibrium of forces on a differentially small element of the length ds for steady flow. The current velocity is given by its vector $[v_i]$, see fig. 1. Considering a static equilibrium configuration, we get the non-linear differential equation 1.

$$\left(\frac{dF_i}{ds} + \frac{dR_i}{ds} + q_i\right) \cdot ds = 0\,. \qquad (1)$$

The vectors $[F_i]$, $[R_i]$ and $[q_i]$ represent the tension, the hydrodynamic load due to viscosity and vorticity of the fluid and the weight of rope per meter in water, respectively. Equation 1 is only valid if the sum of the elements in brackets is zero component-by-component. $dF_i/ds, dR_i/ds$ and q_i are defined as follows:

$$\frac{dF_i}{ds} = \frac{d}{ds}\left(F \cdot \frac{dx_i}{ds}\right) = \frac{dF_i}{ds} \cdot \frac{dx_i}{ds} + F \cdot \frac{d^2x_i}{ds^2} \qquad (2)$$

$$\frac{dR_i}{ds} = \frac{1}{2} \cdot \rho \cdot v^2 \cdot d \cdot c_i \qquad (3)$$

$$[q_i]^T = [0,0,-q] \qquad (4)$$

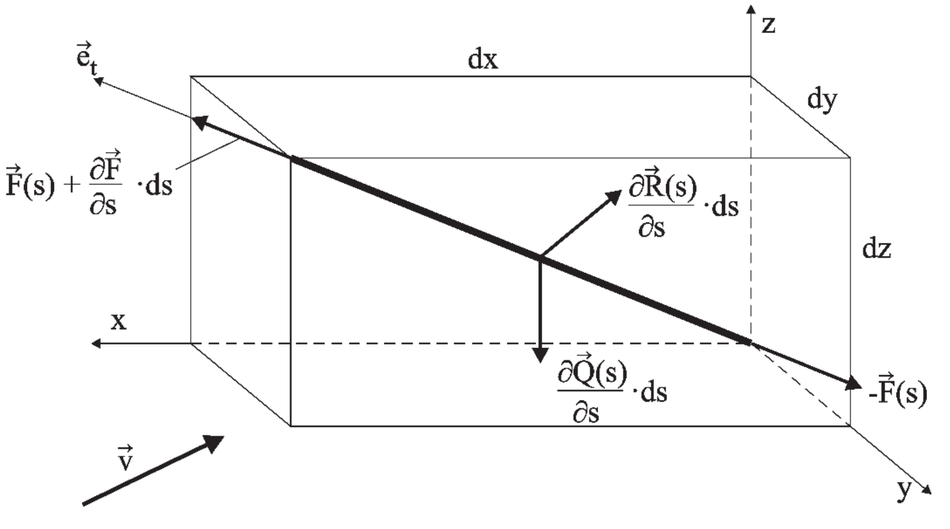

Figure 1: Equilibrium of forces on a differentially small element

Equation 2 is a consequence of the lack of stiffness of the ideal thread. The equation 3 represents a common form for the description of the hydrodynamic load. The directional hydrodynamic coefficients c_i are to be determined in the following.

To solve the differential equation 1, a geometric condition is additionally required. This is given by the sum of squares of the direction cosines according to equation 5

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1\ . \quad (5)$$

The derivation of equation 5 with respect to the coordinate s yields

$$\frac{dx}{ds} \cdot \frac{d^2x}{ds^2} + \frac{dy}{ds} \cdot \frac{d^2y}{ds^2} + \frac{dz}{ds} \cdot \frac{d^2z}{ds^2} = 0\ . \quad (6)$$

As explained initially, an explicit solution of the system of equations is not known, yet, because of the non-linear components of the equations. The equations lead to an inertial value problem which needs to be solved by numerical integration over s.

2.2. Simplified model and results

Sometimes, explicit solutions are easier to handle. Likewise, an explicit solution can be used to validate a numerical one. In this case, we need to simplify the equations physically meaningful to obtain an explicit solution.

Basis of the following analysis is the already known differential equation 1. However, we assume that $q = 0$ because of the balance of weight and buoyancy of the thread in seawater. Furthermore, we consider a flow-loaded flexible thread whose hydrodynamic load is tangentially negligible to its longitudinal axis. That means, the component of the hydrodynamic load perpendicularly to the thread tangent is the only external force. We call this component hydrodynamic normal force $[F_{N,i}]$. It is located by definition in the plane which is spanned by two vectors - the vector of current $[v_i]$ and the vector of the thread tangent $[e_{t,i}] = [dx_i/ds]$.

If v_i and e_i are known one can determine the velocity perpendicularly to the thread tangent using the right-hand rule two times. We get

$$v_{N,i} = \varepsilon_{ijk}\, \varepsilon_{klm}\, e_{t,j}\, v_l\, e_{t,m} = (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, e_{t,j}\, v_l\, e_{t,m}\ . \quad (7)$$

Equation 7 delivers the magnitude of the normal velocity.

$$v_N = \sqrt{v^2 - (v_i\, e_{t,i})^2}. \quad (8)$$

The magnitude of the normal speed is largely dependent on the inner product of $[v_{N,i}]$ and $[e_{N,i}]$ below the root in equation 8. If $[v_{N,i}]$ and $[e_{N,i}]$ oriented parallel to each other v_N disappears. Since the hydrodynamic load disappears in this case, too, this application is outside the range of validity of the mathematical model. If the direction of flow is perpendicularly to the cable the inner product of $[v_{N,i}]$ and $[e_{N,i}]$ disappears. In this case is $v_N = v$. Therefore, v_N is in the interval $0 < v_N \leq v$.

According to Wieselsberger [4], the hydrodynamic drag coefficient of a smooth circular cylinder c_{d-90} in planar cross flow is only dependent on the Reynolds number Rn_d, see fig. 2. Reynolds number of a circular cylinder is defined by

$$Rn_d = \frac{v_N \cdot d}{\nu}\ . \quad (9)$$

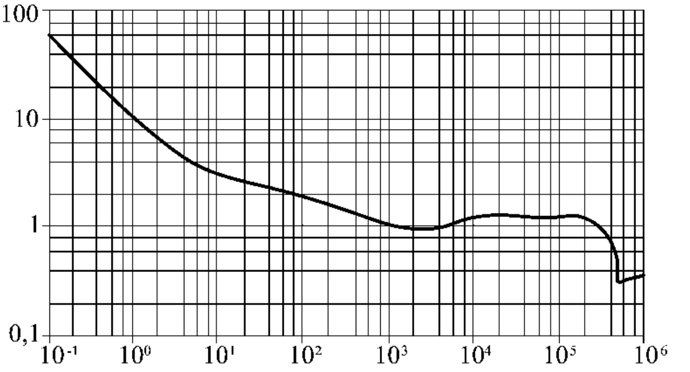

Figure 2: Hydrodynamic drag coefficient c_{d-90} of a smooth circular cylinder with a planar cross flow as a function of Reynolds number Rn_d, see Wieselsberger [4]

Thus, the derivative dR_i/ds according formula 3 can be simplified as follows:

$$\frac{dR_i}{ds} = \frac{1}{2}\, \rho \cdot \sqrt{v^2 - (v_j \cdot e_{t,j})^2} \cdot v_{N,i} \cdot d \cdot c_{d-90}(Rn_d)\ . \quad (10)$$

As long as we can expect a constant flow c_{d-90} remains constant.

We assume without any loss of generality that the current with the velocity v is in direction of the positive x-axis (remember, we set $q = 0$ so the solution is not influenced by gravity). Thus, the normal velocity can be indicated according to equation 7 as follows

$$\begin{pmatrix} v_{N,x} \\ v_{N,y} \\ v_{N,z} \end{pmatrix} = v \cdot \begin{pmatrix} 1 - (e_{t,x})^2 \\ -e_{t,x} \cdot e_{t,y} \\ -e_{t,x} \cdot e_{t,z} \end{pmatrix} \quad (11)$$

Based on this definition, we compute the gradient of the components of the hydrodynamic load and place them accordingly in equation 1. Then, we extend the x-, y- and z-component of the vector equation with $e_{t,x}$, $e_{t,y}$ and $e_{t,z}$, respectively. As a result of a subsequent addition of these three equations we get $dF/ds = 0$ because of equations 6 and 12

$$\frac{dR_i}{ds} \cdot \frac{dx_i}{ds} = 0\ . \quad (12)$$

The integration of $dF/ds = 0$ yields $F = F_0 = \text{const.}$

The result shows that the rope tension is constant under the influence of a hydrodynamic force perpendicular to the thread tangent regardless of its geometrical curve. This result is not surprising. An ideal flexible thread is by definition unable to absorb forces which act transversely to its tangent. A force acting transversely to the thread tangent does not contribute to change the tension. However, the hydrodynamic normal force affects the geometry, i.e., the deflection of the flow-loaded flexible thread.

The calculation of the shape of the flow-loaded thread can be reduced to a two-dimensional problem because the cable is not twisted due to the assumption made for the hydrodynamic load. We will focus on an analysis of the deflection in the $x - 0 - z$ plane, i.e., we consider only the $x-$ component of the vector equation 1. By appropriate mathematical transformations we get the following relationship

$$\frac{de_{t,x}}{ds}+\frac{1}{p}\cdot\sqrt{1-\left(e_{t,x}\right)^2}\cdot\left(1-\left(e_{t,x}\right)^2\right)=0\,, \qquad (13)$$

where the constant p is defined with

$$\frac{1}{p}=\frac{\rho\cdot v^2\cdot d\cdot c_{d-90}}{2\cdot F_0}. \qquad (14)$$

To solve the differential equation 13 we effect the substitution

$$u=\sqrt{1-\left(e_{t,x}\right)^2}$$

and get after mathematical rearrangement

$$\int_{s0}^{s} ds = p\cdot\int_{u0}^{u}\frac{1}{\pm\sqrt{1-u^2\cdot u^2}}du\,. \qquad (15)$$

After integration and subsequent withdrawal of substitution follows

$$s-s_0=\mp\left[\frac{e_{t,x}}{\sqrt{1-\left(e_{t,x}\right)^2}}-\frac{e_{t,x}(x_0)}{\sqrt{1-\left(e_{t,x}(x_0)\right)^2}}\right]\cdot p\,. \qquad (16)$$

To clarify the algebraic sign in equation 16 following regulations we have to observe that:

$$e_{t,x}(x=x_0)\begin{cases} >0 \;\; if\; v_x>0, \text{ minus sign} \\ <0 \;\; if\; v_x<0, \text{ plus sign}\end{cases}. \qquad (17)$$

We define that the lower limit of integration represents the lower thread end. It is displayed with index ’0’ which characterized the origin of the coordinate system, too. That means:

$$x_0=0;\; z_0=0;\; s(x_0,z_0)=s_0\;=0.$$

The integration of equation 16 can only take place in compliance with condition 17:

Case 1: $e_{t,x}(x_0)>0$

The relating formula 16 is as follows:

$$s=-\left[\frac{e_{t,x}}{\sqrt{1-\left(e_{t,x}\right)^2}}-c_0\right]\cdot p\,. \qquad (18)$$

and

$$c_0=\frac{e_{t,x}(x_0)}{\sqrt{1-\left(e_{t,x}(x_0)\right)^2}}\,. \qquad (19)$$

After mathematical rearrangement of equation 18 we get

$$\frac{dx}{ds}=-\frac{c_0\cdot p-s}{\sqrt{p^2+(c_0\cdot p-s)^2}}\,. \qquad (20)$$

Integration provides after separation of variables

$$x=-\left[\sqrt{p^2+(s-c_0\cdot p)^2}-p\cdot\sqrt{1+c_0^2}\right]\,. \qquad (21)$$

Since $x(s)$ is known one can start to identify $z(s)$ by formula 5

$$\left(\frac{dz}{ds}\right)^2=1-\frac{(c_0\cdot p-s)^2}{p^2+(c_0\cdot p-s)^2}=\frac{p^2}{p^2+(c_0-s\cdot p)^2}\,. \qquad (22)$$

The result of the integration is

$$z=p\cdot\left[\operatorname{arsinh}\left(\frac{s-c_0\cdot p}{p}\right)-\operatorname{arsinh}(-c_0)\right]\,. \qquad (23)$$

Case 2: $e_{t,x}(x_0)<0$

In this case, the flow is directed towards the positive x-axis. Thus, only the sign change in a few results. Therefore, it is not necessary to repeat the derivation. We get

$$s=+\left[\frac{e_{t,x}}{\sqrt{1-\left(e_{t,x}\right)^2}}-c_0\right]\cdot p \qquad (24)$$

and

$$x=+\left[\sqrt{p^2+(s-c_0\cdot p)^2-p\cdot\sqrt{1+c_0^2}}\right] \qquad (25)$$

Equation 23 remains unchanged.

Fig. 3 shows results of an example of use. The parameters were chosen as follows:
velocity $v=2[m/s]$, diameter of cable $d=28[mm]$ as well as density of fluid $\rho=1000[kg/m^3]$. Tension F_0 and the thread tangent vector $e_{t,i}(x_0,z_0)$ were successively varied.

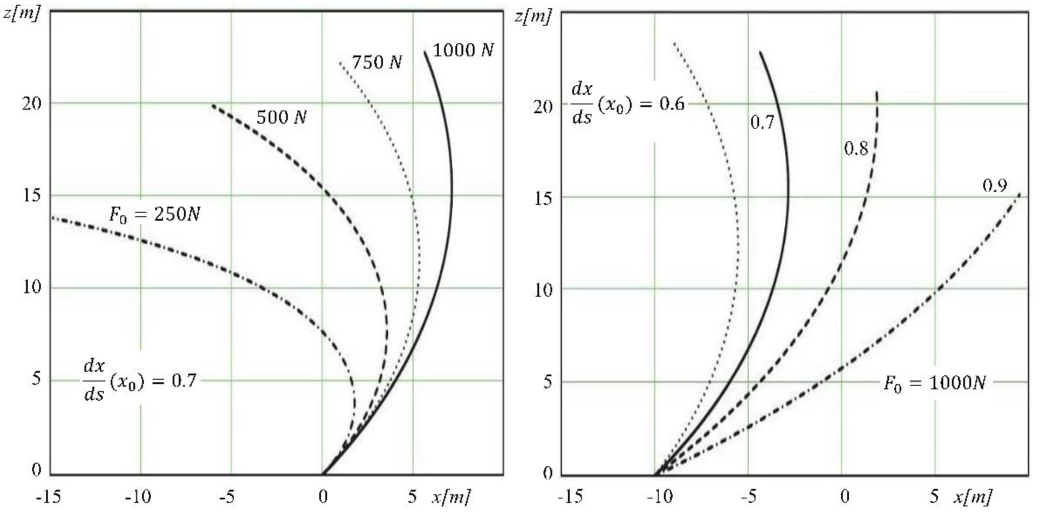

Figure 3: Results of sample calculations at varying initial conditions

3. CONCLUSION

The catenary is so far the only explicit solution of the differential equations describing ideal thread. The concept presented here leads to a second exact solution. This allows the calculation of weight-compensated threads to in uniform flow. This approach can be used directly for the calculation of rope and net systems in fisheries and aquaculture. It can also be used as a basis for the validation of discrete approaches to solutions.

REFERENCES

Faltinsen, O.M. (1990): Sea loads on ships and offshore structures. Cambridge ocean technology series, Cambridge University Press Cambridge, New York, Port Chester, Melbourne, Sydney, 1990

Hamel, G. (1949): Theoretische Mechanik - eine einheitliche Einführung in die gesamte Mechanik (Theoretical Mechanics - an unified introduction into the entire mechanics). Springer Verlag Berlin, Göttingen, Heidelberg 1949

Stengel, H. (1968): Beitrag zur Berechnung von Kurrleinenform und -spannung (Contribution to the calculation of shape and tension of trawl warps). PhD-Thesis, University of Rostock, 1968

Wieselsberger, C. (1921): Neuere Feststellungen über die Gesetze des Flüssigkeits-und Luftwiderstands (Recent findings on the laws of fluid and air resistance). Physikalische Zeitschrift 22 (1921), p. 321