International Journal of Science and Technology Volume 5 No. 7, July, 2016

An Improved Disk-Counting Algorithm for Estimating Dimension of IFS-Based Fractal Image

Salau Tajudeen Abiola O., Bodija Yusuf, Tella Olawale Ismail, Quadri Olatunji Oluwakemi
Department of Mechanical Engineering, University of Ibadan, Nigeria

ABSTRACT

Fractal shapes are characterized using fractal dimension and has application in image quality assessment, texture segmentation, shape classification, data mining, and graphic analysis in many disciplines. A previous research established that the disk-counting method produces better results relative to analytical fractal dimension than the box counting method in estimating the fractal dimension of selected fractals generated using the Iterated Function System (IFS). However, this study presents better estimated disk dimension for IFS-based fractal images based on an improved disk-counting method. The improved algorithm was verified on twenty (20) fractal images generated using appropriate IFS-functions. Although the algorithm takes longer simulation time relative to its counterpart, it was however found to be less error prone and therefore highly recommended whenever accuracy and reliability are needed.

Keywords: *Fractal, Fractal Dimension, Disk-Counting Method, Iterated Function System*

1. INTRODUCTION

Fractals are more than just stunning visual effects — they open up new ways to model natural objects such as river networks, lightning bolts and earthquakes (Lesmoir-Gordon et al, 2010). Fractal geometry makes it possible to describe these objects by their complex shapes and sizes. Fractals often exhibit self-similarity and non-integer value, and can be quantified by a fractal dimension (FD). The concept of FD can be applied in image quality assessment, texture segmentation, shape classification, data mining, and graphic analysis in many disciplines (Ju and Lam, 2009; Jian Li et al., 2009).

Fractals can be generated using various techniques such as iterated function system (IFS), L-systems, escape-time fractals, random fractals and finite subdivision rules. The IFS (originally proposed by John E. Hutchinson in 1981 and later made popular by Michael Barnsley) is a technique for generating fractals which uses affine functions represented by a matrix to generate images that are self-similar. Examples of fractals that can be generated using the IFS technique are the cantor maze, fern , Koch curve, Sierpinski triangle and the terdragon curve (Draves and Reckase, 2003; Fasshauer, 2012; Jepp, 2009; Kumari and Nalini, 2014; Leck, 2015).

Fractal based analysis are ideal for image characterization and have been used for many applications. Surface fractal characteristics of preferential flow patterns in field soils: evaluation and effect of image processing (Susumu Ogawa et al, 1999). The authors used fractal geometry which is increasingly applied to the characterization of fluid pathways via image analysis, leading to the determination of mass and surface FDs. The results show that the authors' choices made during the analysis of stained pattern, caused the FD estimates to vary between 1.32 to 1.64, this made them conclude that stained pattern is indeed a genuine fractal surface; because the dimensions showed no tendency to tend to an integer value. Watanabe et al in 2005 simulated a method on fractal distribution of cracks in rocks using acoustic emission monitoring. The team developed a method using the fractal property of acoustic emission (AE) source distribution to construct a feasible crack distribution model. Results showed that micro-cracking activity, which occurred prior to failure, differed depending on the rock type. This technique was also applied to a micro-seismic data set obtained during mining at an underground Australian coal mine.

Different methods have been proposed to estimate the FD. Annadhason in 2012 reviewed and stated these methods: walking-divider method, box counting method, prism counting method, epsilon-blanket method, perimeter-area relationship, fractional Brownian motion, power spectrum method and the hybrid method. The disk and box methods are among the most popular techniques for estimating the fractal dimension of computable fractals in dimensional Euclidean space. The major reason attributed to this popularity is relative ease of implementation (Salau and Ajide, 2012a).

In estimating the FD of selected fractals generated using the IFS, Salau and Ajide in 2012a established that the disk-counting method produces better results relative to analytical FD than the box counting method. However, there are drawbacks of the algorithm proposed for the disk-counting method, such as higher FD estimates relative to analytical dimension values, and this occurs because the algorithm only considered selecting five (5) solution points of a fractal image at random. Therefore, this study presents an algorithm that utilizes the entire solution points for accurate FD estimates of IFS-based fractal images.

The consistency of numerical results obtained using twenty (20) selected IFS-based fractal images justified its benefits.

This paper is structured as follows. In Section 2, fractal image generation using IFS, the fundamental concept of FD and the improved disk-counting algorithm are presented. In Section 3, twenty (20) selected fractal images generated using appropriate IFS-functions were evaluated using the proposed disk-counting estimation. Section 4 draws the conclusions.

2. BASIC DEFINITION AND DISK-COUNTING METHOD

An improved disk-counting algorithm was developed and coded in FORTRAN (Formula Translation) to estimate the FD of IFS-based fractals. The detailed algorithm can be summarized as follows.

1. Generate IFS-based fractals using well-suited affine functions.
2. Select individual points on the IFS-based fractal image as a pivot. Thereafter, determine the optimum count of disks required to cover the image entirely at different observation scales, and compute the fractal disk dimension.
3. Distribute the estimated FDs for each point on the IFS-based fractal image.

2.1. IFS procedure

The basic principle to generate fractals using IFS is based on randomly selecting affine functions to compute fractal image solution points. Affine transformation projects a point (x, y) in the plane to another point (x', y') and generally represented as

$$w_i \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix} \tag{1}$$

where a, b, c, d, e and f are coefficients of an affine transformation, and $|a_i d_i - b_i c_i| < 1$ (Adcock et al., 2003; Fasshauer, 2012; Jepp, 2009; Mulder, 2015; Natoli, 2012; Ng and Cohen, 1994).

The IFS procedures used in the present study to generate fractals are outlined below:

1. Adopt from literature, suitable affine transformations for a fractal image of choice.
2. Assign selection probabilities to each transformation. That is, assign a number between 0 and 1 to each transformation that determines the likelihood of the transformation being selected randomly at any given iteration.
3. Plot the resulting point (x_1, y_1) in the plane.
4. Select another transformation at random (according to the assigned probability) and plot the new image of the point (x_1, y_1) under the selected transformation. Label this as point (x_2, y_2).
5. Repeat this process as large as 5000 times to produce a sequence of solution points: (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)...(x_{5200}, y_{5200}).
6. Trade off the first 200 transient solution points to produce steady solution points which results in the generation of a well refined fractal image.

2.2. Basic concept of the new box-counting algorithm and FD estimation

Fractal dimension describes how an object changes with a measuring scale (Falconer, 2003). The FD of fractals adopted from (Salau and Ajide, 2012a) can be computed using power related functions. Hence,

$$Y = CX^{d} \tag{2}$$

$$Log(Y) = Log(C) + dLog(X) \tag{3}$$

$$y = k + dx \tag{4}$$

Y = Minimum number of disks of known size required to cover wholly a fractal image.

C = Constant of proportionality.

X = Minimum number of disks of known size to cover datum length (known length).

d = Slope (FD).

Equation 3 is rewritten into equation 4 i.e.

$y = Log(Y)$

$$k = Log(C)$$

$$x = Log(X)$$

A log-log regression between measuring scales of disk length observed (X) and the corresponding optimal count of covering disks (Y) is estimated to find the slope d, which is the FD.

This study used the disk-counting method to estimate the FD of fractals generated using IFS. The new approach proposed that makes use of the entire solution points of a fractal for accurate FD estimates is stated below:

1. The image is mapped on a 2-dimension Euclidean space, such that a point is specified using two coordinates.
2. Determine the characteristics length of the attractor, which is the farthest distance between coordinate points of the image.
3. The farthest distance represents the diameter of the disk required to cover the image entirely.
4. Find the disk radius and reduce it per scale of observation.
5. Pick the first coordinate point of the image as the pivot point for disk count observation.
6. Find the distance between the pivot point and all other points of the image.
7. If the distance between the pivot point and another point is less or equal to the value of the disk radius for a scale of observation, then a disk count is recorded.
8. If the distance between the pivot point and another point is more than the value of the disk radius for a scale of observation, another pivot is chosen randomly and step 5 is repeated for another disk count (meaning more than 1 disk will be required to cover the image totally) till the points are captured.
9. Repeat step 5 for each point of the image as pivot point.
10. Iterate results five times for ten different scales of observation.
11. The disk count with the least value (optimal count of disks) for 5 different iterations was recorded for varying scales of observation.
12. Evaluate disk dimension for each pivot points using the power law described in equation (2-4).

3. Results and Discussions

To illustrate the relationship between FD and image quality, and compare the accuracy and consistency of different disk-counting algorithms, twenty (20) fractals with suitable IFS functions were extracted from (Jepp, 2009; Salau and Ajide, 2012a; Salau and Ajide, 2012b; Salau and Ajide 2013; http://hiddendimension.com/fractalmath/ifs_fractals_main.html; http://soft.vub.ac.be/~tvcutsem/teaching/wpo/grafsys/ex4/les4.html).

3.1. Numerical analysis - disk dimension estimation of the Sierpinski triangle

Using the proposed algorithm to estimate the fractal disk dimension at each pivot point, the control parameters required include the disk count and scale of observation. In order to determine the choicest count of disks required to cover a fractal image completely, the least number of disks from five (5) successive iterations was selected for each scale of observation and listed in table 1.

Table 1: Optimal count of disks required in covering the Sierpinski triangle

Scale of Disk Diameter AB Covering a Sierpinski Triangle = X	Minimum Number of Disks Required to Cover the Sierpinski Triangle = Y	Logarithm of X	Logarithm of Y
1	3	0.0000	1.0986
2	8	0.6931	2.0794
3	14	1.0986	2.6390
4	20	1.3863	2.9957
5	27	1.6094	3.2958
6	35	1.7917	3.5553
7	43	1.9459	3.7612
8	53	2.0794	3.9702
9	68	2.1972	4.2195
10	75	2.3026	4.3175

Table 1 represents the least number of disks recorded for the first coordinate center point for a 500 point image of the Sierpinski triangle. Fig. 1 shows linear interpolation on a set of points between the log of measurement scale (X) and the log of disk count (Y) for the Sierpinski triangle of 500 points.

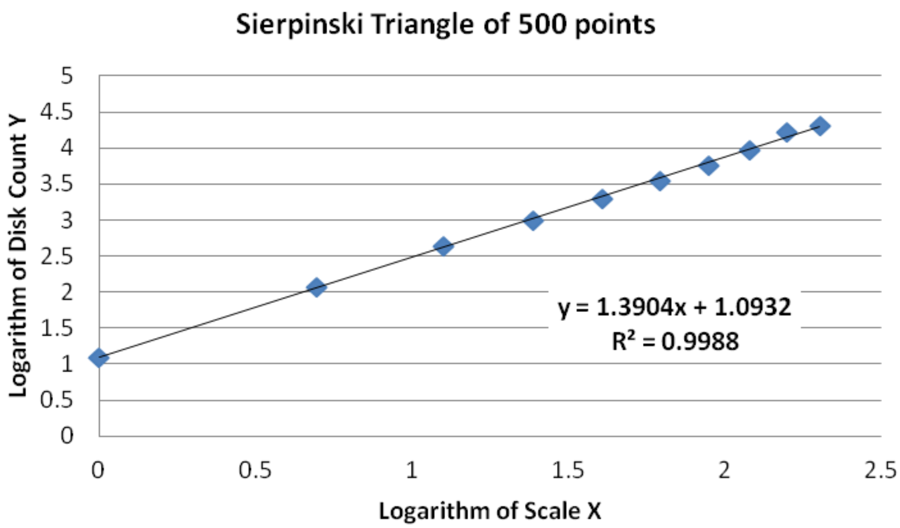

Fig. 1: Linear interpolation showing the disk dimension at the first coordinate pivot point.

The slope of the line determined represents the disk dimension of the fractal at the first coordinate pivot point, and shown in Fig. 1 to have a value of 1.3904.

Fig. 2 plots the mean disk dimension using the proposed algorithm at different fractal image points. It can be observed that the disk dimension consistently changes with increasing number of image points. This implies that the quality of an image gradually increases when more image points are considered. To visualize the effect of increasing image points on image quality, various sample points of the fractal image was generated using IFS and displayed in Fig 3.

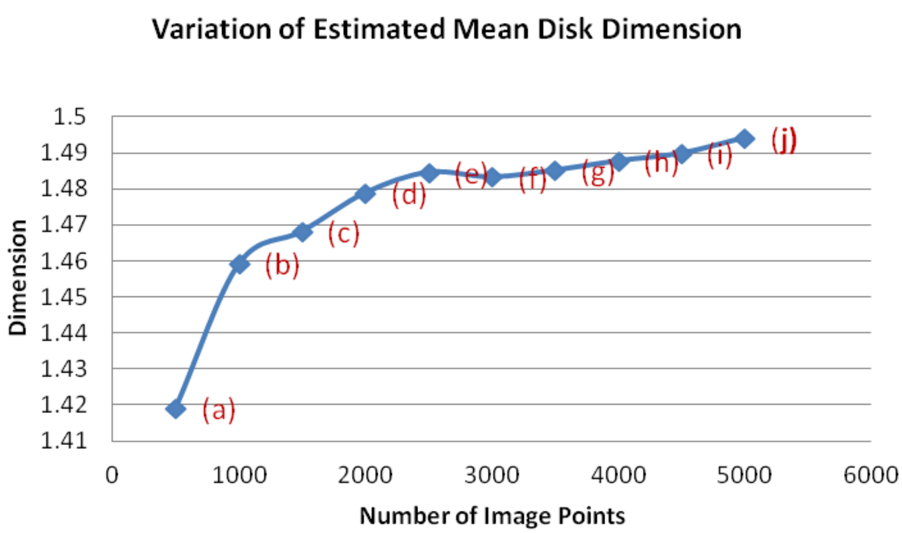

Fig. 2: Estimated mean disk dimension variation range of the Sierpinski triangle computed using the new algorithm with different values of image solution point

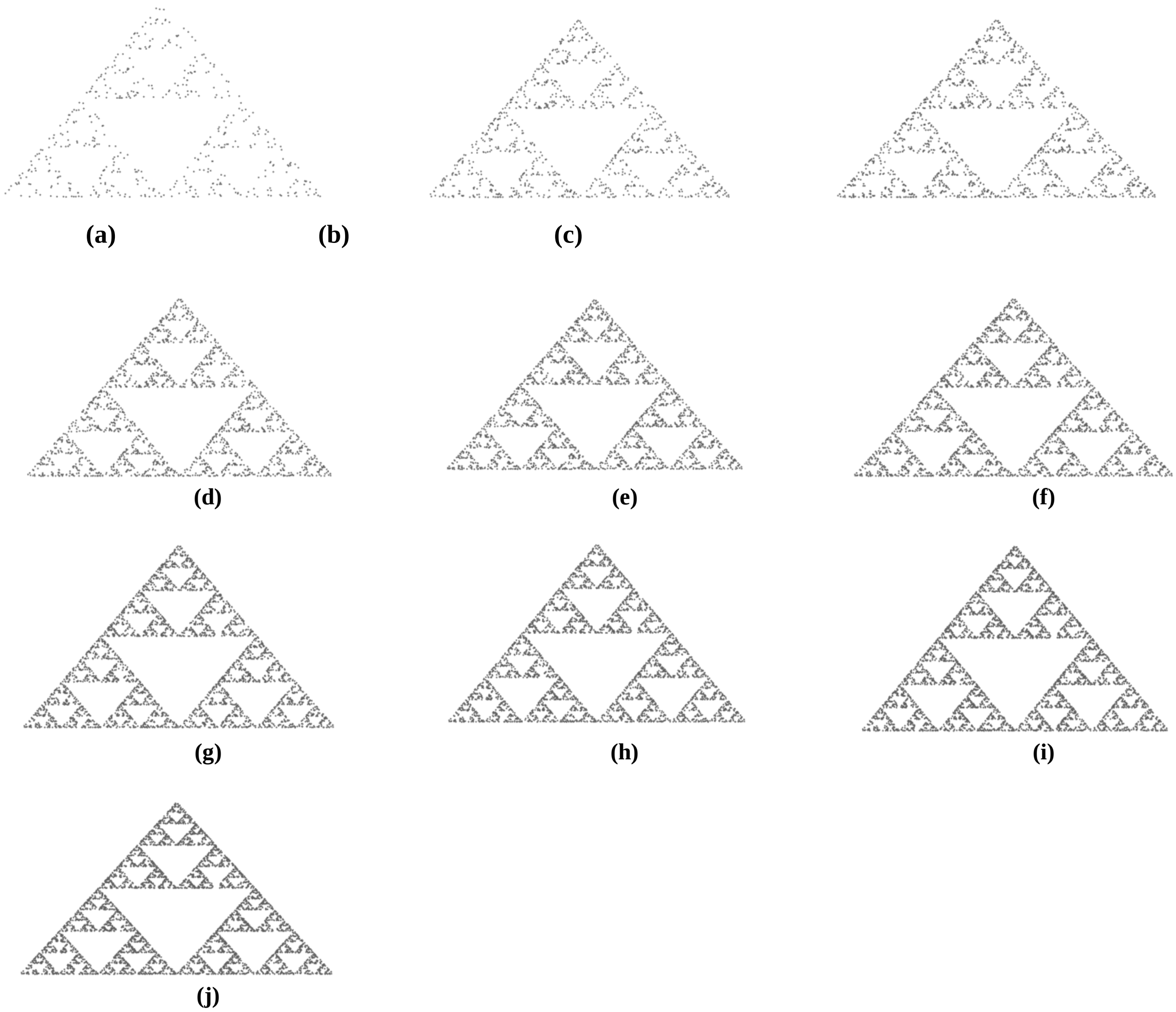

Fig. 3: The effect of increasing number of image points on image quality using the Siepinski triangle: (a) Sierpinski triangle with 500 points; (b) Sierpinski Triangle with 1000 points; (c) Sierpinski Triangle with 1500 points; (d) Sierpinski Triangle with 2000 points; (e) Sierpinski Triangle with 2500 points; (f) Sierpinski Triangle with 3000 points; (g) Sierpinski Triangle with 3500 points; (h) Sierpinski Triangle with 4000 points; (i) Sierpinski Triangle with 4500 points; (j) Sierpinski Triangle with 1000 points.

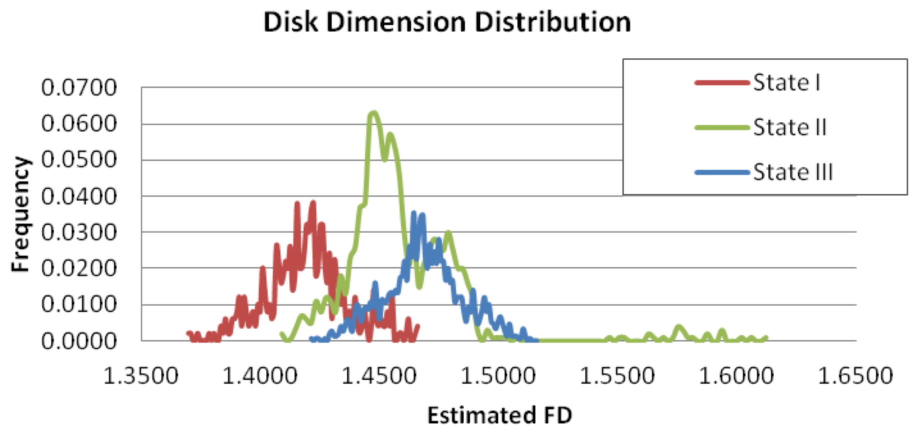

Fig. 4: Disk dimension distribution with increasing solution points: (i) State I = Sierpinski triangle with 500 points; (ii) State II = Sierpinski triangle with 1000 points; (iii) State III = Sierpinski triangle with 1500 points. The corresponding plots are highlighted in Fig. 2.

The result of the fractal image with 500 points (Fig. 3a) is blurred which indicates less quality (lowest disk dimension) while that of 5000 points (Fig. 3j) is the brightest (highest disk dimension). Visually, fig. 3 indicates that the disk dimension is responsive to increasing image points. To further illustrate the effect of increasing image points on estimated FD, the cumulative frequency for three image states and estimated FDs were plotted in fig. 4. It is noted that all image states (fig. 4) has multiple frequency peaks in a region corresponding to their estimated FD, and there is a gradual shift towards the theoretical dimension (Sierpinski triangle = 1.5850 or log3/log2) with increasing solution points. Hence, the proposed algorithm can be interpreted as reliable to predict results of estimated FDs at varying fractal image points.

3.2. Comparison between analytical and estimated FDs of IFS-based fractals

To examine the accuracy and consistency of two different disk-counting algorithms, analytical FD values of IFS-based fractals were extracted from (salau and ajide, 2013; https://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension) and compared with estimated FD results derived from the proposed algorithm and that of Salau and Ajide (2012a).

The mean disk dimension across varying fractal image points plotted in Fig. 5 shows a result trend of two different disk-counting algorithms. It can be noted that as the fractal image (Sierpinki triangle) point increases; results from the proposed algorithm rose steadily, tending towards a limit, while that of the previous algorithm was unstable. Thus, the proposed algorithm can be interpreted as a stable and consistent FD estimator.

To further compare the accuracy and stability in results from both algorithms, mean FD estimates for twenty (20) IFS-based fractals were listed and analyzed in table 2.

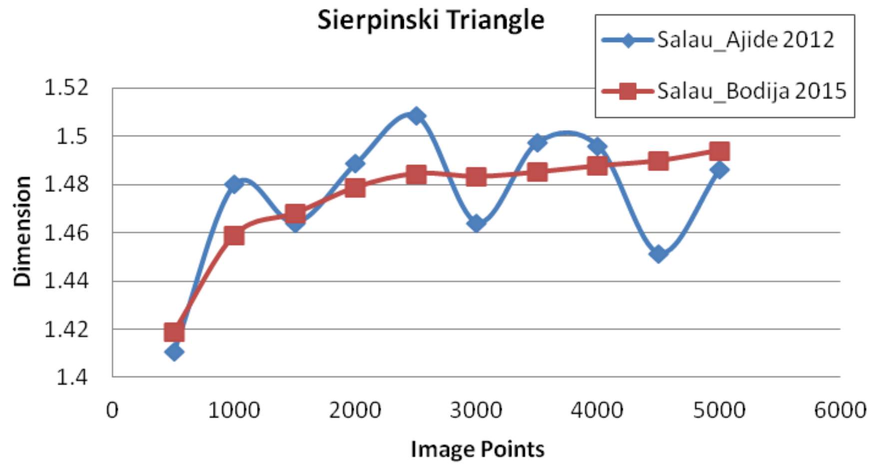

Fig. 5: Estimated mean disk dimension variation range of the Sierpinski triangle computed using the previous (Salau_Ajide 2012a) and proposed algorithm (Salau_Bodija 2015) with different values of fractal image point.

As tabulated in table 2, estimated FD results from the proposed algorithm were consistently lower than their corresponding analytical value with an absolute percentage relative error range from 0.9 to 19.4, and this is lower than that of the previous (Salau and Ajide, 2012a) algorithm from 1.2 to 19.5 (66.7% more accurate than its counterpart).

Table 2: Analytical and Estimated Fractal Dimension of Twenty Fractals

S/N	Fractals	Fractal dimension			Absolute percentage relative error (%)		Remark (%error: is the proposed algorithm lower than the previous?)
		Analytical	Previous algorithm	Proposed algorithm	Previous algorithm	Proposed algorithm	
1	Sierpinski Triangle	1.5850	1.4862	1.4942	6.2	5.7	Yes
2	Terdragon Curve	2.0000	1.7176	1.7086	14.1	14.6	No
3	Cantor Maze	N/A	1.4902	1.4833	--	--	--
4	Twin Xmas Tree	N/A	1.4943	1.4894	--	--	--
5	Twig	1.2619	1.0413	1.0172	17.5	19.4	No
6	Weed	N/A	1.3737	1.3631	--	--	--
7	Weierstrass Function	1.5000	1.2081	1.2204	19.5	18.6	Yes
8	Fractal L	N/A	1.4481	1.4683	--	--	--
9	Koch	1.2619	1.1096	1.1125	12.1	11.8	Yes
10	Fern	N/A	1.5846	1.5367	--	--	--
11	Maple Leaf	N/A	1.7086	1.7112	--	--	--
12	Pine Tree	N/A	1.5937	1.4737	--	--	--
13	Fractal T	N/A	1.5447	1.4679	--	--	--
14	Sierpinski Hexagon	1.6309	1.6511	1.6165	1.2	0.9	Yes
15	Tree	1.5849	1.3766	1.4040	13.1	11.4	Yes
16	Vicsek Fractal	1.4649	1.3974	1.4062	4.6	4.0	Yes
17	The Devil's Staircase	N/A	1.6192	1.5744	--	--	--
18	A Crystal-like Structure	N/A	1.7072	1.5649	--	--	--
19	Sierpinski Carpet	1.8928	1.7901	1.6829	5.4	11.1	No
20	Chaos Image	N/A	1.5187	1.4877	--	--	--

Note. N/A = No data available at the time of preparing this report.

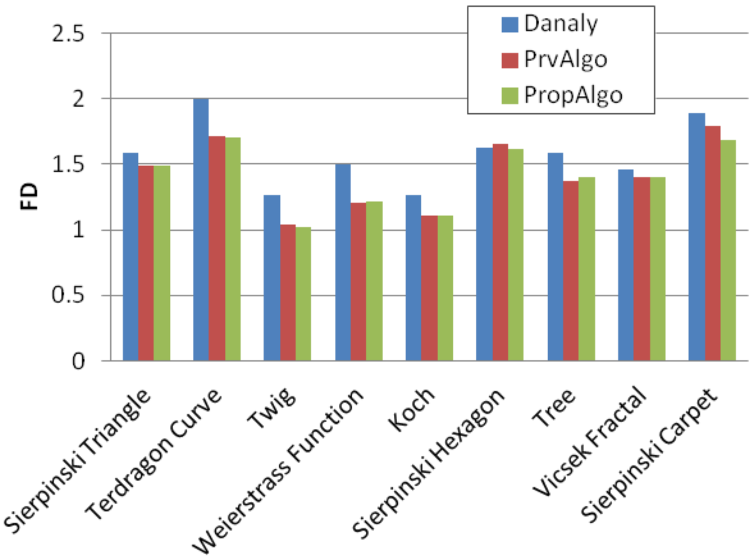

Figure 6: Comparison between estimated fractal dimension of selected fractals using the previous and proposed algorithm: Danaly = analytical dimension; PrvAlgo = previous algorithm; PropAlgo = proposed algorithm.

In addition, Fig. 5 presents a comparison between estimated and analytical FD values for selected fractals obtained from table 2, and a closer look at these results provides further insights. It can be observed that the previous algorithm had an unstable estimation and yielded the largest deviation from analytical value for the Sierpinski hexagon. The proposed algorithm consistently produced much smaller estimated FD than analytical FD values for all fractals considered. This further confirms that the proposed algorithm is more accurate at estimating the FD of IFS-based fractal images.

4. CONCLUSIONS

An improved disk-counting algorithm is proposed to estimate the FD of IFS-based fractals. To improve accuracy of estimates, three major contributions were highlighted in this study and described as follows: the first one is to use appropriate affine functions to generate fractals with the IFS method; the second lies in estimating fractal dimension using each coordinate point of a fractal image as the algorithm's pivot, as this ensures all points in the image is considered during the estimation; and the last is to distribute estimated fractal dimension for each point of the IFS-based fractal.

Using 20 fractals generated with well-suited IFS-functions, the proposed disk-counting algorithm was compared with the previous disk-counting algorithm. It was found out that the previous algorithm took less simulation time relative to the proposed algorithm, but for a certain fractal (Sierpinski hexagon); it overestimated the FD, which yielded a value exceeding the theoretical dimension. On the other hand, the proposed algorithm consistently yielded more accurate results than the previous one. Subsequently success rate for estimates from the proposed algorithm was 66.7% higher than its counterpart and therefore less error prone.

Based on these results, it is recommended that the proposed disk-counting algorithm should be used where accurate and stable fractal dimension estimates are required. The proposed algorithm, although takes longer simulation time; can easily be programmed on any platform following its procedures. Furthermore, it can be employed in categorizing images (shape classification), image quality assessment (image characterization), texture segmentation, and in artistic/beautification works.

REFERENCES

Angus Leck (2015). Iterative Function Systems (IFSs) and the Multiple Reduction Copy Machine (MRCM). Retrieved 6 June 2015 from http://www.maths.usyd.edu.au/u/UG/IM/MATH2916/r/Leck_Presentation.pdf

Annadhason A. (2012). Methods of fractal dimension computation. IRACST - International Journal of Computer Science and Information Technology & Security (IJCSITS), 2(1), 1-4.

Bruce Adcock, Youri Dimitrov, Gerald Edgar, Dean Eiger, Charles Estill, Larry Lindsay, Steven Miller, James Tyne (2003). Fractal Working Group. Retrieved 20 May 2015 from http://www.williams.edu/Mathematics/sjmiller/public_html/math/generalmath/classes/fractal.pdf

Christopher Natoli (2012). Fractals as fixed points of Iterated Function Systems. Illinois, USA: University of Chicago.

Greg Fasshauer (2012). Introduction to the profession: random affine transformations and fractals in MATLAB. Illinois, USA: Illinois Institute of Technology.

Henk Mulder (2015). Derivative coordinates for analytic tree fractals and fractal engineering. Retrieved 10 June 2015 from http://ariv.org/pdf/1501.01675

Jian Li, Qian Du, Caixin Sun (2009). An improved box-counting method for image fractal dimension estimation. Pattern Recognition, 42(11), 1224-1233.

Kennetth Falconer (2003). Fractal geometry. New York, USA: Wiley.

Ng W.L. and Cohen H.A. (1994). Theoretical Basis of Block-Oriented Fractal Image Coding. Melbourne, Australia: Technical report, La Trobe University.

Nigel Lesmoir-Gordon, Arthur C. Clarke, Benoît Mandelbrot, David Pennock, Gary Flake, Ian Stewart, Michael Barnsley, Will Rood (2010). The colours of infinity. The beauty and power of fractals. London, United Kingdom: Springer-Verlag London limited.

Pauline Jepp (2015). Self similarity and fractal geometry. Retrieved 8 July 2015 from http://www.pages.cpsc.ucalgary.ca/~jacob/Courses/Winter2002/CPSC601-73/Slides/Fractals.pdf

Raja Kumari K. and Nalini C. (2014). Improvement of image quality based on fractal image compression. Middle-East Journal of Scientific Research, 20(10), 1213-1217.

Salau T.A.O. and Ajide O.O. (2012a). Disk and box dimensions: selected case studies of fractals with IFS codes. International Journal of Science and Technology, 1(5), 1-14.

Salau T.A.O. and Ajide O.O. (2012b). Linear superposition of symmetric IFS-based attractors and fractal characterization. International Journal of Engineering and Technology (IJET), 2(12), 1-7.

Salau T.A.O. and Ajide O.O. (2013). Numerical transformations and fractal characterization of some selected IFS-based fractals. International Journal of Science and Technology, 4(2), 1-13.

Scott Draves and Erik Reckase (2008). The fractal flame algorithm. Retrieved 30 June 2015 from http://www.flam3.com/flame.pdf

Susumu Ogawa, Philippe Baveye, Charles W. Boast,Jean-Yves Parlange, Tammo Steenhuis (1998). Surface fractal characteristics of preferential flow patterns in field soils: evaluation and effect of image processing. Geoderma, 88(9), 109–136.

Watanabe Yoshinori, Itakura Ken-Ichi, Sato Kazuhiko, Fujii Yoshiaki, Balasu Rao, Guo Hua, Luo Xun, (2005). A modeling method of fractal distribution of cracks in rocks using AE monitoring. Acoustic Emission Group, 23, 119-127.

Wenxue Ju and Nina S.N. Lam (2008). An improved algorithm for computing local fractal dimension using the triangular prism method. Computers and Geosciences, 35(11), 1224-1233.