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Mathematical Model for Economic Packaging Setup Frequency in a Multi Item, Time Proportional Demand Inventory System, With Storage Limitation

Delphine.Suliy. Ghormo, A. Tahir, Ibrahim Isa Adamu & A. A. Momoh
Department of Mathematics,
Moddibo Adama University of Technology,
Yola, Nigeria

ABSTRACT

This work is aimed at developing a mathematical model for multi item inventory system incorporating storage limitation. In deriving the model, we adopted compartment modeling approach. The model equation derived can be used to determine the optimal cost of maintaining inventory with limited storage facility for manufactured and packaged products. The result generated, hypothetical case study, shows that joint replenishment for multi item inventory system with storage limitation incurs extra cost than same inventory system without storage limitation this is in line with inventory practice because, storage limitation attracts additional cost due to lost sales and good will from customers, except if backlogging is allowed or no shortages are allowed. Thus in an inventory system with storage limitation without backlogging, our model is recommended

Keywords: *Joint Replenishment, Multi Item, Holding Cost, Storage Limitation, Time Proportional Demand*

1. INTRODUCTION

Many companies order a group of items simultaneously, rather than individually. This is known as joint replenishment. The principal concept behind joint replenishment is that several items are ordered from a single /multiple supplier/suppliers respectively, and several products share the same means of transportation. Joint replenishment is used in multi item inventory system in order to optimize the total cost when ordering Multi items. The economic advantage of manufacturing these items jointly and then packaging them individually is due to the fact that if items are replenished (manufactured and packaged) individually then each item is accountable for the full set up cost for each of its manufacturing and packaging runs. When a product is packaged in more than one type of container size immediately after manufacture, these items (each representing a particular type or size of container) are jointly replenished (Goyal, 1980). The problem of determining the frequency of packaging set ups for the jointly replenished items is akin to the problem of determining the economic ordering frequency of items procured from a single supplier (Starr and Miller, 1962).

Silver (1976), worked on the problem of determining the economic packaging setup frequency for jointly replenished items,He assumed among other things that the items are packaged into only one brand and the demand for the packaged items is constant throughout the planning horizon. He developed a model for determining the optimum packaging setup frequency for jointly replenished items in a multi item, single brand inventory system with static demand. The model developed is tedious to implement as it requires a lot of mathematical computations. Hall (1988), proposed a simple heuristic for the Multi-Item Replenishment and Storage Problem (MIRSP), where all items share a common replenishment interval. For simplicity's sake, we call the policy proposed by Hall; the Rotation Cycle (RC) policy for the MIRSP. Hall provides, first, a detailed schedule of all the items-order epochs in a given order interval, and second, the order interval value that brings the overall average cost to a minimum. Hariga (1988), in his extensive work, provides some solution techniques for the same problem and some of its variants. These methods, which are based both on exact formulations and heuristics, are not shown to exhibit any ex-ante bound on the optimality gap.

Anily (1991) considered an infinite-horizon, multi-item replenishment problem. In addition to the usual setup and holding costs incurred by each item, an extra charge proportional to the peak stock volume at the warehouse is due. This last cost raises the need for careful coordination while making decisions on the individual item order policies. They restrict to the class of policies that follows a stationary rule for each item separately. They derive a lower bound on the optimal average cost over all policies in this class and investigate the worst case of the Rotation Cycle policy. Depending on the problem's parameters, the Rotation Cycle policy may yield an extremely good solution but in other settings this heuristic may generate an extremely poor policy. They also developed a new heuristic whose performance is at least as good as that of the Rotation Cycle procedure.

Ibrahim (2014) developed a mathematical model for multi-item inventory system incorporating shortages; however he did not consider the possibility of storage (ware house) limitations when ordering the items. The reality is that, sometimes manufacturers are faced with the problem of limited storage facility due to demand stimulated production; hence the need for extending the work by Ibrahim (2014) to incorporate storage (ware house) limitation becomes necessary. In this paper, we consider a Multi-item inventory system with storage limitation and infinite planning horizon.

2. MATERIALS AND METHOD

2.1. Assumptions

The following assumptions are made in developing the model.

(i) Storage facilities (storage space) constraint is imposed.
(ii) Demand for each item is a linear function of time.
(iii) Shortages are not allowed.
(iv) Manufacturing and packaging setups are undertaken at equal intervals of time.
(v) Cost of manufacturing setups is allocated equally to all packaged items irrespective of the number packaged.
(vi) Time horizon is infinite.
(vii) Minimization of the average cost is the objective.
(viii) Time is treated as a discrete variable and divided into sub intervals of unit duration each, such that a day is unit duration.4

2.2. Notations

The following notations were made.

Symbol	Meaning
S	Cost of manufacturing set up for the *m* items
N	Frequency of manufacture set up for the *m* items
m	Number of items
$t_i\,(i=1,2,\ldots 366)$	The discretized time units for which (t_i, t_i+1) is of unit duration.
For the j^{th} item $j=1(1)m$	
$f_j(t)=a_j+b_j t$	Demand at time t, where a_j is the demand rate at $t=0$ and b_j is the trend value of the demand
s_j	Cost of a packaging set up
h_j	Annual unit stock holding cost
y_j	Storage area requirement per manufactured item
k_j	Ratio of frequency of manufacturing setups to frequency of packaging setups (the ratio must be of the form $1:1$, $2:1$ etc.)
$k(k_1,k_2,k_3,\ldots,k_m)$	Is a vector
Y	Maximum available storage area for all m items
λ	Lagrange multiplier

3. DERIVATION OF THE MODEL

The total cost model equation $CU(N,\underline{K})$ for the m items based on the component modeling approach is given as;

Cost function

$CU(N,\underline{K})$ = Holding cost + manufacturing cost + packaging setup cost

Constraint

Storage limitation.

Now for the j^{th} item;

Stock holding cost component

The annual demand is $\sum_{i=1}^{366} f_j(t_i)$, hence the average stock is $\frac{\sum_{i=1}^{366} f_j(t_i)k_j}{2N}$.

And consequently, the stock holding cost is $\frac{\sum_{i=1}^{366} h_j f_j(t_i)k_j}{2N}$.

Packaging setup cost component

Cost of packaging setup is $\frac{N s_j}{k_j}$.

Manufacturing setup cost component

Allocated cost of manufacturing setup is: NS/m

Storage constraint component

Total products manufactured and packaged for across the m items is $\sum_{j=1}^{m} y_j N_j$

The storage constraint demands that $\sum_{j=1}^{m} y_j N_j \le Y$, where, $N_j > 0$ and Y are the manufacturing setups for the j^{th} item and the maximum available storage area for all m items respectively.

Hence the total average cost $CU(N,\underline{K})$ for the m package items is given by

$$CU(N,\underline{K}) = \frac{1}{2N}\sum_{j}^{m}\sum_{i=1}^{366} f_j(t_i)h_i k_i + N\sum_{j=1}^{m}\left(\frac{S}{m}+\frac{s_j}{k_i}\right)\text{...(A)}$$

Subject to $\sum_{J=1}^{M} y_j N_j \le Y$

With variables and parameters as defined above

Determination of k_j

Given the value of N, the economic value of $k_j = k_j(N)$ for the j^{th} item must satisfy the following conditions:

$$CU(N, K_j(N)) + CU(N, K_j(N)+1) \le 0 \quad \text{.......(1)}$$

$$CU(N, K_j(N)) + CU(N, K_j(N)-1) \le 0 \quad \text{.......................................(2)}$$

From (1) and (2) we get

$$K_j(N)(K_j(N)-1) \le N^2\left(\frac{2s_j}{\sum_{i=1}^{366} f_j(t_i)h_j}\right) \le K_j(N)(K_j(N)+1) \quad \text{.. (3)}$$

Equation (3) can be used to determine the optimal value of $k_j(N)$ given the (optimal) value of N, we now set out to determine the optimal value of N.

Determination of N and λ

Under the assumption of no shortage, the mathematical model representing this inventory situation is given as:

Minimize

$$CU(N,\underline{K}) = \frac{1}{2N}\sum_{j}^{m}\sum_{i=1}^{366} f_j(t_i)h_i k_i + N\sum_{j=1}^{m}\left(\frac{S}{m}+\frac{s_j}{k_i}\right)$$

Subject to

$$\sum_{J=1}^{M} y_j \, N_j \quad \le \quad Y \text{ where,}$$

$$N_J \quad > \quad 0, \quad j \quad = \quad 1,2,...m\,.$$

The lagrangean function is formulated as

$$L\left(\lambda, N_j, N_2 ... N_n\right) \quad = \quad CU\left(N_j, N_2 ... N_n\right) \quad - \quad \lambda \sum_{j=1}^{m}\left(y_j \, N_j \, - Y\right)$$

Where $\lambda\left(<0\right)$ is the Lagrange multiplier

The optimal values of N_j and λ can be obtained from;

$$\frac{dl}{dN} = N\sum_{i=1}^{366}\left(\frac{S}{m}+\frac{s_j}{k_j}\right) - \frac{1}{2N^2}\sum_{i=1}^{366}\sum_{i=1}^{366} f_j\left(t_i\right)h_j\,k_j\,\lambda\, y_i \quad = \quad 0\cdot$$

……………………………………………….(B)

$$\frac{dl}{d\lambda} \quad = \quad \sum_{i=1}^{366} y_j \, N_j \, + Y_j \quad = \quad 0 \,\cdot$$

From *(B)* we have:

$$N\left(k\right) = \left[\frac{\sum_{j=1}^{m}\sum_{i=1}^{366} f_j\left(t_i\right)h_j\,k_i}{2\sum_{j=1}^{m}\left(\frac{S}{m}+\frac{s_j}{k_j}\right) \; - \; \lambda y_j}\right]^{\frac{1}{2}} \qquad (4)$$

Substituting (4) into (A) and simplifying, we get;

$$CU\left(N, \underline{K}\right) \quad = \quad \left[2\sum_{j=1}^{m}\sum_{i=1}^{366} f_j\left(t_i\right)h_j\,k_j\sum_{j=1}^{m}\left(\frac{S}{m}+\frac{s_j}{k_j}\right) - \lambda y_j\right]^{\frac{1}{2}}$$

(5)

4. RESULT AND DISCUSSION

4.1. RESULT

In this work we have developed a mathematical model for a multi item joint replenishment with storage limitation. It is clear from the expressions above that; constrained optimal values for N, K cannot be determined without the knowledge of λ, similarly N cannot be determined without the knowledge of K and vice-versa. A number of heuristic methods are available for solving such type of problems.

4.1.1. The Heuristic

Here we wish to use the modify the heuristic used by Ibrahim (2014) by augmenting two additional steps to approximate an unconstrained value for the space requirement for the *j*th item as follows;

STEP 1: Compute the unconstrained value of the space requirement according to

$$\left[\frac{\sum_{i=1}^{366} f_j\left(t_i\right)h_j}{2\left(\frac{S}{m}+s_j\right)}\right]^{\frac{1}{2}} \qquad \ldots (6)$$

STEP 2: Check if the unconstrained optimal values satisfy the storage constraint. If it does,stop; the solution is optimal. Otherwise, use the constraint equation, to compute the value, given by;

$$N^* = \left[\frac{\sum_{i=1}^{366} f_j\left(t_i\right)h_j}{2\left(\frac{S}{m}+s_j\right) - \lambda y_i}\right]^{\frac{1}{2}} \qquad (7)$$

The formula shows that N^* is dependent on the value of λ^*. For $\lambda = 0$, N^* gives the unconstrained solution. The value of λ^* can be found in the following manner: because by definition $\lambda < 0$ for the minimization case, we successively decrement λ by a reasonably small amount, and use it in (7) to compute the associated N^*. The desired λ^* yields the values that satisfy the storage constraint $\sum_{J=1}^{M} y_j \, N_j \; \le \; Y\,\cdot$

STEP 3: Determine the starting value $N1$ of N from

$$N1 = \left[\sum_{i=1}^{366} f_s(t_i) h_s / 2(S + S_s)\right]^{\frac{1}{2}} - - - - - - -(9)$$

Where $\left[\sum_{i=1}^{366} f_s(t_i) h_s / 2(S + S_s)\right]^{\frac{1}{2}} = \max_j \left[\sum_{i=1}^{366} f_j(t_i) h_j / 2(S + S_j)\right]^{\frac{1}{2}}$

STEP 4: Determine the vector $\underline{K}$ from (3)

STEP 5: Determine N from (4)

STEP 6: If N (or $\underline{K}$) has not changed, calculate CU(N,$\underline{k}$) using (5) and stop else go to STEP 3

4.2. DISCUSSION

A close look at the demand function, given by $f_j(t) = a_j + b_j t$, suggest that the model may be used for inventory systems with constant demand (*in a joint replenishment environment*) incorporating storage limitation. This can be deduced when the trend value $b_j = 0$, is used in the model. We augment discussion by applying our model to a hypothetical example, in this case, we adopt the hypothetical example used by Ibrahim (2014), with little modification by imposing a storage limitation, as follows;

Consider an inventory system where S = ₦2, m = 4 and the data given in **table I** below. Supposed the available storage space for the items manufactured is Y=250fts. We wish to determine the optimum cost CU(N,$\underline{k}$).

Table 1: Cost of packaging setup holding cost and demand function for 4 items

j	s_j	h_j	$f_j(t)$
1	8	1	$\frac{1}{10} + t$
2	2	1	$\frac{1}{3} + t$
3	10	1	$1 + \frac{t}{5}$
4	6	1	$\frac{1}{5} + \frac{3t}{2}$

In other words if the optimal quantity of manufactured and packaged goods is greater than $250\,fts$ then the constraint has an effect on the solution of the problem and can be obtained from (6). This is a theoretical work based on hypothetical values but closed to real life problems.

SOLUTION".

Using the heuristic used by Ibrahim (2014), we have the computed model components values as contained in **table II** below;

Table 2: Computed model components using values in table I

J	$\sum_{i=1}^{366} f_j(t_i)$	$\sum_{i=1}^{366} f_j(t_i) \times h_j$	y_j	$\left[\sum_{i=1}^{366} f_j(t_i) \times h_j / 2(\frac{S}{m} + s_j)\right]^{\frac{1}{2}}$
1	66831.6	66831.6	1	62.7
2	33497.3	33497.3	1	81.9
3	13725.0	13725.0	1	25.6
4	100265.7	100265.7	1	87.8

Now, the unconstrained optimal values for the space requirement can be obtained as

$$\left[\frac{\sum_{j=1}^{4} \sum_{i=1}^{366} f_j(t_i) h_j}{2\left(\frac{S}{m} + s_j\right)}\right]^{\frac{1}{2}} = \quad 258\,units$$

Since N^* (258) is more than $250\,fts$ we can say that the storage capacity is limited, and hence we use $N^* = \left[\frac{\sum_{i=1}^{366} f_j(t_i) h_j}{2\left(\frac{S}{m} + s_j\right) - \lambda y_i}\right]^{\frac{1}{2}}$ .to obtain the approximate constraint value for the space requirement by iterating on the value of λ < 0 until we obtain the value of λ that satisfies $\sum_{J=1}^{M} y_j N_j \leq Y$. With series of iterations, the value of λ that satisfies $\sum_{J=1}^{M} y_j N_j \leq Y$ falls between $-0.3572 > \lambda > -0.6572$. Now, using λ = *-0.6752*, we obtained the following Computed constrained model components using values in table I.

Table 3 Computed constrained model components using table I and Using $\lambda = -0.6572$

j	$\sum_{i=1}^{366} f_j(t_i)$	$\sum_{i=1}^{366} f_j(t_i)\times h_j$	y_j	$\left[\frac{\sum_{i=1}^{366} f_j(t_i)h_j}{2\left(\frac{s}{m}+s_j\right)-\lambda y_j}\right]^{\frac{1}{2}}$
1	66831.6	66831.6	1	61.5
2	33497.3	33497.3	1	76.9
3	13725.0	13725.0	1	25.2
4	100265.7	100265.7	1	85.7

The constrained optimal values are

$$\left[\frac{\sum_{j=1}^{4}\sum_{i=1}^{366} f_j(t_i)h_j}{2\left(\frac{s}{m}+s_j\right)-\lambda y_j}\right]^{\frac{1}{2}} = 249.3\, units$$

Since constrained optimal $249.3\, units$ is less than $250\, fts$, this implies that; the storage constraint is satisfied. Next, we determine $N1$,

STEP 3: Since Max$_j$

$$\left[\frac{\sum_{i=1}^{366} f_s(t_i)h_s}{2(s+s_s)-\lambda y_j}\right]^{\frac{1}{2}} = 85.7$$

corresponds to item 4 in the above table, the starting value N1 of N would be computed with regard to the fourth item. N1 as computed according to equation (7) is $N1 = 78$

STEP 4: Determine $k_j(N_1)$ such that expression (3) holds for $j = 1(1)m$, thus we have

$$k_1(N_1)=1,\quad k_2(N_1)=1,\quad k_3(N_1)=3,\quad k_4(N_1)=1,$$

and hence obtaining

$$k(N_1)=(1,1,3,1)$$

STEP 3: Computing the next value of N that is N_2 according to equation (5), we have $N_2 = 75$

STEP 5: Determining a new least integer for $k_j(N_2)$ such that expression (3) holds for

$j = 1(1)m,$

$$k_1(N_2)=1,\quad k_2(N_2)=1,\quad k_3(N_2)=3,\quad k_4(N_2)=1$$

and hence obtaining $k(N_2)=(1,1,3,1)$.

Since

$k(N_1) = k(N_2)$, i.e. k has not changed, we calculate $CU(N,\underline{K})$ according to equation (6). Using $N_2 = 75$ and $k(N_2) = 1,1,3,1$ for the items to compute the associated cost, $CU(N,\underline{K})$, we have $CU(N,\underline{K}) = N3236$

4.2. Conclusion

From the above result, we can see that we have extended the work by Ibrahim (2014) to incorporate storage limitation. Thus, when confronted with stock management in a multi-item single brand time proportional demand under limited storage warehouse, our model should considered. Besides the fact that the model has been designed to solve inventory problems for linearly increasing demand, it can be used for inventory systems with constant demand (in a joint replenishment environment) as can be deduce if $b_j = 0$, we have $f_j(t_i) = a_j$.

REFERENCES

Goyal, S.K. (1980), Determination of economic packaging frequency in a multi-brand joint replenishment inventory system *European Journal of Operational Research* vol. 4, pp 185-188.

Hall, N.G. (1988), A multi item EOQ Model with Inventory Cycle Balancing. *Naval Res. Logis*. 35, 319-325.

Ibrahim Isa Adamu (2014), Mathematical Model for economic packaging set up frequency in a multi item time, proportional demand inventory system; *ICASTOR Journal of Mathematical Sciences* vol 8, pp. 13 -20

Silver E.A. (1976), A simple method of determining order quantities in Joint replenishments under deterministic demand *Management Science* Vol. 22. USA.

Starr, M.K. and Miller, D.W. (1962), Inventory Control: *Theory and practice,* prentice –Hall, Inc., Englewood Cliffs, 104.