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Dynamical Consistence of Euler Scheme for Leslie-Gower Predator-Prey Model with Holling Type II Functional Response

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ABSTRACT

In this paper, we investigate Euler scheme for Leslie-Gower predator-prey model with Holling-type II. Dynamical behavior of the discrete model such as the existence and stability of the equilibrium point are analyzed. Investigation on the dynamical consistence of the discrete model against the continuous model, which includes the existence condition and stability of the equilibrium points has been performed. It is shown that the discrete model is dynamically consistent with its continuous model for relatively small step-size.

Keywords: *Discrete Leslie-Gower Predator-Prey Model, Holling Type II Functional Response, Stability, Dynamical Consistent.*

1. INTRODUCTION

It is known that Leslie-Gower model is more realistic than Lotka-Volterra model because the carrying capacity of predator population in Leslie-Gower predator-prey model depends on prey density. However, in recent years, many modifications and studies are constantly conducted to accommodate some biological phenomena. In 2003, Aziz-Alaoui and Okiye worked on a Leslie-Gower predator-prey model by using Hollling type II as functional response. They investigated boundedness of solutions, positive invariant exixtence and global stability of coexist equlibrium points by constructing such a Lyapunov function. Then in 2006, Nindjin et al studied Leslie-Gower predator-prey model with time delay. They also analyzed boundedness of solutions and global stability of equilibrium points by constructing a Lyapunov function.

In 2012, Yu review the model researched by Aziz-Alaoui and Okiye that is a Leslie-Gower predator-prey model with Holling Type II functional response because there are several condition of being scant just in a theorem of global stability .Yu fix two necessary condition to global asymptotic stability of a positive equilibrium point by fluctuation Lemma and by Lyapunov method.

Predator-prey models can be formulated as discrete time models. Huang et al. studied predator-prey model, Wu and Li studied predator-prey model with Hassel-Varley functional response, Selvam et al. studied predator-prey model with prey harvesting, Naji and Lafta studied predator-prey model with ratio-dependent functional response, Wang et al. studied predator-prey model with Alle effect, Agiza et al.studied predator-prey model with Holling Type II functional response, Fayeldi studied SIR model with nonmonotone incidence rate. Discrete model from Euler method may show rich dynamics compared to the continuous one in Jie et al.

2. MATHEMATICAL MODEL

In 2003, Aziz-Alaoui and Okiye worked on a Leslie-Gower predator-prey model by using Holling type II functional response:

$$\frac{dx}{dt} = \left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right)x,$$
$$\frac{dy}{dt} = \left(r_2 - \frac{a_2 y}{x + k_2}\right)y,$$

where $x(t)$ be the population density of prey and $y(t)$ be the population density of predator at time t. All parameters assuming only positive values. These parameters are defined as follows: r_1 is the growth rate of prey, r_2 describes the growth rate of predator. a_1 is the maximum value which per capita reduction rate of prey can attain, b_1 measure the strength of competition among individuals of species prey, a_2 has a similar meaning to a_1. k_1 measure the extent to which environment provides protection to prey , k_2 measure the extent to which environment provides protection to predator.

In the present work, applying the forward Euler scheme to system (1), we obtain the discrete time Leslie-Gower predator prey model with Holling type II functional response as follows:

$$x_{n+1} = x_n + hx_n\left(r_1 - b_1 x_n - \frac{a_1 y_n}{x_n + k_1}\right)$$
$$y_{n+1} = y_n + hy_n\left(r_2 - \frac{a_2 y_n}{x_n + k_2}\right)$$

where h is the step size.

3. DYNAMICAL ANALYSIS

Equilibrium point of system (2):

$E_0 = (0,0), E_1 = \left(\frac{b_1}{r_1}, 0\right), E_2 = \left(0, \frac{r_2 k_2}{a_2}\right)$ and (x^*, y^*) which satisfies equation $r_1 - b_1 x^* - \frac{a_1 y^*}{x^* + k_1} = 0$ and $r_2 - \frac{a_2 y^*}{x^* + k_2} = 0$.

From equation $r_2 - \frac{a_2 y^*}{x^* + k_2} = 0$, we get $y^* = \frac{r_2(x^* + k_2)}{a_2}$. If y^* substitution to equation $r_1 - b_1 x^* - \frac{a_1 y^*}{x^* + k_1} = 0$, we get $r_1 - b_1 x^* - \frac{a_1 r_2 (x^* + k_2)}{a_2 (x^* + k_1)} = 0$.

One can easily see that satisfies

$$Ax^{*2} + Bx^* + C = 0, \tag{3}$$

Where $A = b_1 a_2 > 0, B = b_1 a_2 k_1 + a_1 r_2 - a_2 r_1$, and $C = a_1 r_2 k_2 - a_2 r_1 k_1$. Equation (3) can have at most two positive solutions, and hence system (2) can have at most two positive equilibrium point. Precisely, we have the following five cases

Case 1: If $C < 0$, then system (2) has unique interior positive equilibrium point

$$E_{3,1} = (x^*_{3,1}, y^*_{3,1}) = \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}, \frac{r_2(x^*_{3,1} + k_2)}{a_2}\right)$$

.

Case 2: If $B < 0$ and $D = B^2 - 4AC = 0$, then system (2) has unique positive equilibrium point

$$E_{3,2} = (x^*_{3,2}, y^*_{3,2}) = \left(\frac{-B}{2A}, \frac{r_2(x^*_{3,2} + k_2)}{a_2}\right).$$

Case 3: If $C = 0$ and $B < 0$, then system (2) has unique positive equilibrium point

$$E_{3,3} = (x^*_{3,3}, y^*_{3,3}) = \left(\frac{-B}{A}, \frac{r_2(x^*_{3,3} + k_2)}{a_2}\right).$$

Case 4: If $B < 0, C > 0$ and $D = B^2 - 4AC > 0$ then system (2) has two positive equilibrium point :

a. $$E_{3,4+} = (x^*_{3,4+}, y^*_{3,4+}) = \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}, \frac{r_2(x^*_{3,4+} + k_2)}{a_2}\right)$$

b. $$E_{3,4-} = (x^*_{3,4-}, y^*_{3,4-}) = \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}, \frac{r_2(x^*_{3,4-} + k_2)}{a_2}\right)$$

Case 5: If no condition in case 1-4 holds, then system (2) has no positive equilibrium point.

To study the stability of equilibrium points of the model, we first give the useful Lemma (in Xiaoli and Xiao), which can be easily proved by the relations between roots and coefficients of a quadratic equation

Lemma 3.1. Let $F(\lambda) = \lambda^2 - p\lambda + q$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. Then

1. $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only $F(-1) > 0$, and $q < 1$,
2. $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
3. $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$, and $q > 1$,
4. λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $p^2 - 4q < 0$ and $q = 1$.

Let λ_1 and λ_2 are two eigenvalues of the equilibrium point. We recall some definitions of topological types for a equilibrium point (x^*, y^*). (x^*, y^*) is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. (x^*, y^*) is called *saddle* if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). (x^*, y^*) is called *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. And $(x^*, y^*)$is called *non-hyperbolic* if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

The Jacobian matrix of (2) at E_0 is

$$J(0,0) = \begin{pmatrix} 1 + hr_1 & 0 \\ 0 & 1 + hr_2 \end{pmatrix}.$$

Hence the two eigenvalues are $\lambda_1 = 1 + hr_1 > 1$ and $\lambda_2 = 1 + hr_2 > 1$. According to Lemma 3.1, we can obtain that the equilibrium E_0 is unstable (*source*).

The Jacobian matrix of (2) at E_1 is

$$J\left(\frac{b_1}{r_1}, 0\right) = \begin{pmatrix} 1 - hr_1 & -\frac{ha_1 r_1}{r_1 + b_1 k_1} \\ 0 & 1 + hr_2 \end{pmatrix}.$$

The two eigenvalues are $\lambda_1 = 1 - hr_1$ and $\lambda_2 = 1 + hr_2 > 1$.

The equilibrium E_1 is *saddle* if $h < \frac{2}{r_1}$ and unstable (source) if $h > \frac{2}{r_1}$.

The Jacobian matrix of (2) at E_2 is

$$J\left(0, \frac{r_2 k_2}{a_2}\right) = \begin{pmatrix} 1 + hr_1 - \frac{ha_1 r_2 k_2}{a_2 k_1} & 0 \\ -\frac{ha_1 r_2^2}{a_{21}} & 1 - hr_2 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1 + hr_1 - \frac{ha_1 r_2 k_2}{a_2 k_1}$ and $\lambda_2 = 1 - hr_2$.

Proposition 3.2: Let $h_1 = \frac{2a_2 k_1}{(a_1 r_2 k_2 - a_2 r_1 k_1)}, h_2 = \frac{2}{r_2}$, and $a_1 r_2 k_2 > a_2 r_1 k_1$.

1. If $h < \min(h_1, h_2)$ then E_2 is a sink.
2. If $\min(h_1, h_2) < h < \max(h_1, h_2)$ the E_2 is a saddle.
3. If $h > \max(h_1, h_2)$ then E_2 is a source.
4. If $h = h_1$ or $h = h_2$ then E_2 is *non-hyperbolic.*

Proof:

The eigenvalues of Jacobian matrix of (2) at E_2 are $\lambda_1 = 1 + hr_1 - \frac{ha_1r_2k_2}{a_2k_1}$ and $\lambda_2 = 1 - hr_2$.

E_2 is a stable (*sink*) if $-1 < 1 + hr_1 - \frac{ha_1r_2k_2}{a_2k_1} < 1$ and $-1 < 1 - hr_2 < 1$.

First condition

$$-2 < h\left(r_1 - \frac{a_1r_2k_2}{a_2k_1}\right) < 0$$

equivalent to

$$0 < \frac{h}{a_2k_1}(a_1r_2k_2 - r_1a_2k_1) < 2$$

Since $a_1r_2k_2 > a_2r_1k_1$, then

$$0 < h < \frac{2a_2k_1}{a_1r_2k_2 - r_1a_2k_1} = h_1$$

Second condition

$$-2 < -hr_2 < 0$$

so

$$0 < h < \frac{2}{r_2} = h_2$$

E_2 is a sink if $h < \min(h_1, h_2)$.

2. Clearly that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) equivalent to $h > h_1$ and $h < h_2$ or $h < h_1$ and $h > h_2$. Hence, E_2 is a *saddle* if $\min(h_1, h_2) < h < \max(h_1, h_2)$.
3. Clearly that E_2 is a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ equivalent to $h > \max(h_1, h_2)$.
4. Clearly that E_2 is *non-hyperbolic* if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ equivalent to $h = h_1$ or $h = h_2$.

The Jacobian matrix of system (2) at (x^*, y^*) is in the form of

$$J(x^*, y^*) = \begin{bmatrix} 1 - hb_1x^* + \frac{hx^*(r_1 - b_1x^*)}{(x^* + k_1)} & -\frac{ha_1x^*}{x^* + k_1} \\ \frac{hr_2^2}{a_2} & 1 - hr_2 \end{bmatrix}$$

Then the characteristic equation is

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = 2 - hb_1x^* + \frac{hx^*(r_1 - b_1x^*)}{(x^* + k_1)} - hr_2,$$

$$q = 1 - hb_1x^* + \frac{hx^*(r_1 - b_1x^*)}{(x^* + k_1)} - hr_2 + h^2b_1x^*r_2 - \frac{h^2x^*r_2(r_1 - b_1x^*)}{(x^* + k_1)} + \frac{h^2a_1x^*r_2^2}{a_2(x^* + k_1)}.$$

Interior equilibrium point (x^*, y^*) case 2 and 4(b) is unstable.

Proposition 3.3

Let

$$B_0 = r_2x^*\frac{(2Ax^* + B)}{a_2(x^* + k_1)},$$

$$B^* = \frac{1}{4}\left(\frac{-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1}{(x^* + k_1)}\right)^2$$

and $h^* = \frac{a_2(-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1)}{r_2x^*(2Ax^* + B)}$,

then

1. Interior equilibrium point (x^*, y^*) case 1, 3, and 4(a) is stable (*sink*) if $-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1 > 0$ and one of the following conditions holds.
   (a). $0 < h < h^*$ and $B_0 \geq B^*$, or
   (b). $0 < h < h_3$ and $B_0 < B^*$.
2. Interior equilibrium point (x^*, y^*) case 1, 3, and 4(a) is a *saddle* if $-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1 > 0$, $h_3 < h < h_4$ and $B_0 < B^*$.
3. Interior equilibrium point (x^*, y^*) case 1, 3, and 4(a) is a *source* if $-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1 > 0$ and one of the following conditions holds.
   (a). $h > h^*$ and $B_0 \geq B^*$, or
   (b). $h > h_4$ and $B_0 < B^*$.
4. Interior equilibrium point (x^*, y^*) case 1, 3, and 4(a) is a *non-hyperbolic* if $-r_1x^* + 2b_1x^{*2} + b_1k_1x^* + r_2x^* + r_2k_1 > 0$ and one of the following conditions holds.
   (a). $h = h^*$ and $B_0 \geq B^*$, or
   (b). $h = h_3$ or $h = h_4$ and $B_0 < B^*$.

5. NUMERICAL SIMULATION

To confirm our previous analytical finding we perform some numerical simulations using discrete system (2). We divide these simulations into three simulation.

The first numerical simulation is done by taking the parameters
$r_1 = 1, r_2 = 3, b_1 = 0.4, a_1 = 0.2, a_2 = 0.3, k_1 = 0.1$ and $k_2 = 0.3$ which satisfy the condition that makes equilibrium point E_2 stable and interior equilibrium point at all case not exist. Then there are three equilibrium points, namely $E_0 = (0,0), E_1 = (0.4,0)$, and $E_2 = (0,3)$. The equilibrium point E_2 is a sink if $h < 0.4$, otherwise it is unstable. The stability of E_2 can be observed from Figure 1, where the trajectories obtained by taking $h = 0.4$ and any initial values are convergent to equilibrium point E_2. Hence equilibrium point E_2 is a topologically sink.

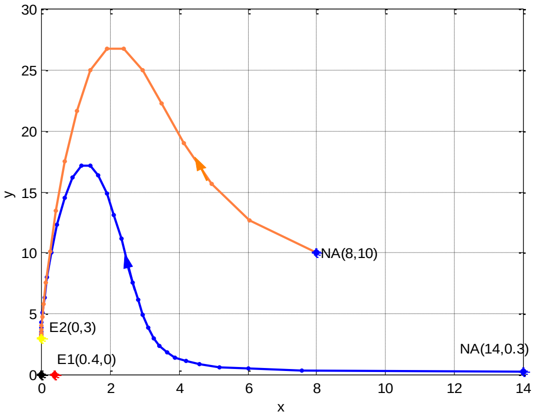

Figure 1. Phase portrait system (2) where E_2 is a sink and interior equilibrium point not exist

The second simulation is done to demonstrate existence of E_0, E_1, E_2 and interior equilibrium point which satisfy case 1, 3, and 4(a) by taking the parameters

$r_1 = 1, r_2 = 0.5, b_1 = 0.1, a_1 = 5, a_2 = 3, k_1 = 0.6$ and $k_2 = 0.3$. Then there are four equilibrium points, namely $E_0 = (0,0), E_1 = (0.1,0), E_2 = (0,0.05)$ and interior equilibrium point (x^*, y^*) which satisfy case 1, 3 and 4(a) $(2.4787, 0.4631)$. The interior equilibrium point (x^*, y^*) is sink if $h < 0.9087$. For illustration we plot the solution using $h = 0.1$ in Figure 2.

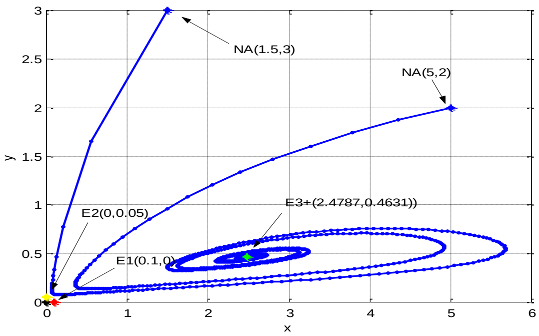

Figure 2. Phase portrait system (2) where interior equilibrium point (x^*, y^*) which satisfy case 1, 3 and 4(a) stable and the interior equilibrium point (x^*, y^*) which satisfy case 2 and 4(b) not exist

The third simulation is done to illustrate existence of E_0, E_1, E_2, and interior equilibrium point which satisfy all case by taking the parameters
$r_1 = 3, r_2 = 1.2, b_1 = 0.9, a_1 = 0.01, a_2 = 5.2, k_1 = 0.0025$ and $k_2 = 10$. Then there are five equilibrium points, namely $E_0 = (0,0), E_1 = (0.3,0), E_2 = (0,2.3077)$, interior equilibrium point (x^*, y^*) which satisfy case 1, 3 and 4(a) $(3.3231, 3.0746)$ and interior equilibrium point (x^*, y^*) which satisfy case 2 and 4(b) $(0.0052, 2.3089)$. The interior equilibrium point (x^*, y^*) which satisfy case 1, 3 and 4(a) is sink if $h < 0.6711$. For illustration we plot the solution using $h = 0.1$ in Figure 3.

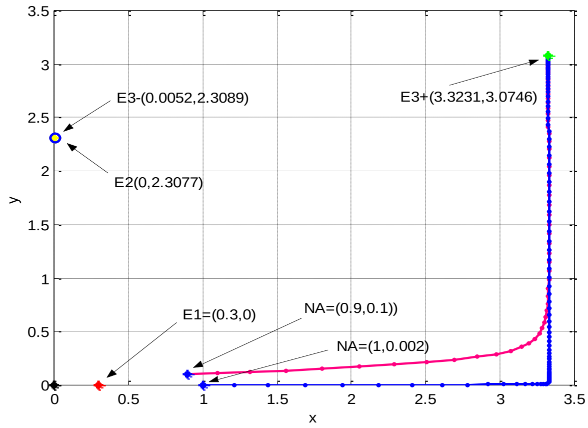

Figure 3. Phase portrait system (2) where interior equilibrium point (x^*, y^*) which satisfy case 1, 3 and 4(a) stable and the interior equilibrium point (x^*, y^*) which satisfy all case exist

5. CONCLUSION

In this paper, we considered a 2-dimensional discrete Lesli-Gower predator-prey model and obtained equilibrium points, stability of equilibrium points and the results are illustrated with suitable parameter values. Numerical simulation are presented to show the dynamical behavior of the system. The equilibrium points are consistent with the continuous model for relatively small step-size.

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