International Journal of Science and Technology Volume 4 No. 3, March, 2015

Block Runge-Kutta Type Method for Direct Integration of Second Order Bvps in Odes

Zamurat A. Adegboye, Ahu A. Adamu & Shehu Usman
Dept. of Mathematics, Statistics and Computer Science,Kaduna Polytechnic,
Kaduna.Nigeria.

ABSTRACT

In this paper, we present the Quade's Type Multistep (QTM) method. The process produces Quade's scheme and some hybrid form which are combined together to form a block method. The method is extended to the case in which the approximate solution to a second order (special or general) Boundary Value Problems(BVPs) can be calculated and is of order six which is A-stable, possesses the Runge-Kutta stability property and has an implicit structure for efficient implementation.

Keywords: *Quade's Type Multistep, Second Order Bvps, A-Stable And Runge-Kutta Stability Property*

1. INTRODUCTION

In an earlier work (Yahaya and Adegboye (2007)),the authors constructed and implemented a new Quade's type four-step block hybrid multistep method for accurate and efficient parallel solution of a first order Ordinary Differential Equations(ODEs),The result converged better to the exact solution with A-stable region of absolute stability Also in Badmus and Adegboye (2010) the authors obtained two different hybrid block schemes of Quade's type from single continuous formulation, and numerical experiments were applied for the purpose of comparison.

This paper is part of research effort aimed toward reformulating for efficient and accurate use of the linear multistep method and Quade's discrete scheme is considered, the standard Quade's (see Lambert (1973)) discrete scheme which is well known but because of the requirement for a starting value, the method is not popular.

An implicit 4-point block Runge-Kutta method have been successfully implemented for the solution of initial value problems of the form

$$y' = f(x,y) \qquad y(x_o) = y \qquad (1.1)$$

$$y'' = f(x,y,y') \qquad y(x_o) = y \quad y'(x_o) = \beta \qquad (1.2)$$

(see Yahaya and Adegboye (2011))

In this work, we will reformulate the QTM method into Runge-Kutta method for the solution of boundary value problem of the form

$$y'' = f(x,y,y') \quad y(a) = \alpha \quad y(b) = \beta \quad x \in [a,b] \qquad (1.3)$$

A number of numerical methods for these classes of problems have been extensively developed. On the contrary the problem of the form (1.3) is not commonly discussed in literature.There are several interrelated aims in the search for such method, such as high order, low error constants, satisfactory stability property such as A-stability , low implementation costs and self starting. We particularly wish to emphasize the combination of a multi-step structure with the use of off-grid points, we seek a method that are multistage and multi-value because it will be convenient to extend the general linear method formulation to the high order Runge – Kutta case (Butcher (2003)) by considering a polynomial.

$$y(x) = \sum_{j=1}^{i-1} \phi_j(x) y_{n+j} + h\sum_{j=1}^{i-1} \varphi_j(x) f(\bar{x}_j, y(\bar{x}_j)) \ldots (1.4)$$

where t denotes the number of interpolation point x_{n+j}, j =0,1--- *t-1*; and m denotes the distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$ j = 0,1---m-1 chosen from the given step $[x_n, x_{n+k}]$. Here y and f are smooth real N-dimensional vector functions. The numerical constant coefficients ϕ_j ,(j = 0,1---t-1) and $h\ \phi_j$, (j = 0,1---m-1) of (1.4) are to be determined since they are selected so that accurate approximations of well behaved problems step size can be a constant or change in the numerical integration process. This method is characterized by Butcher array as follows

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W \end{array} \qquad \ldots.(1.5)$$

For the implicit Runge–Kutta method for the numerical integration of the first order initial value problem (1.1).
While for the implicit Runge–Kutta method for the numerical integration of the second order initial value problem (1.2-1.3) written in Butcher – array form we have

$$\begin{array}{c|c|c} \alpha & A & \bar{A} \\ \hline & b^T & \bar{b}^T \end{array} \qquad A = a_{ij} = \beta \qquad \bar{A} = \bar{a}_{ij} = \beta^2$$

$$\beta = \beta e \qquad b = w \qquad \bar{b}^T = w^T\beta \qquad \text{.........(1.6)}$$

where A denotes the (t+m)*(t+m) real matrix and b and α are real vectors of dimension (t+m) and $\alpha_i \in [0, k]$, ί = 1, 2 --- t+m-1. According to Kulikov (2003), if the matrix A in the Butcher's array is a lower triangular matrix with zero main diagonal, then the method is called explicit.

This paper is part of a research effort to reformulate for efficient and accurate use, the linear multistep method and the Quade's discrete scheme is considered.

The paper is organized as follows in section 2 we show how the QTM method was constructed, this leads to section 3, where the reformulation is discussed, In section 4, some numerical test for the new methods from which the conclusion (section 5) are drawn are presented.

2. CONSTRUCTION OF THE QTM

Consider an approximate solution to (1.1-1.3) in the form of power series

$$y(x) = \sum_{j=o}^{t+m-1} a_j x^j \qquad \text{...(1.7)}$$

$$a \in R, j = o(1)t + m - 1, Y \in C^m(a,b) \subset P(x)$$

$$y'(x) = \sum_{j=o}^{t+m-1} j a_j x^{j-1} \qquad \text{...(1.8)}$$

Where a_j 's are the parameters to be determined, t and m are points of interpolation and collocation. To form our matrix D we collocate (1.8) at x_{n+j}, $j = 0,1,3,4$ and interpolate (1.7) x_{n+j}, $j = 0,1,3$. Specifically, k=4, t=3 and m=4 yield the following system of equations:

$$\sum_{j=o}^{t+m-1} a_j x^j = y_{n+j} \qquad j = 0,1,3,4 \qquad \text{... (1.9)}$$

$$\sum_{j=o}^{t+m-1} j a_j x^{j-1} = f_{n+j} \qquad j = 0,1,3 \qquad \text{... (2.0)}$$

Following the multistep collocation of Yusuph,Y.A. and Onumanyi P (2002), we invert once the matrix D which is of dimension (t+m)*(t+m). The proposed continuous formulation takes the form:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_3(x)y_{n+3} + h\{\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\}$$

When using Maple mathematical software to invert (1.9) and (2.0), obtaining values for $a_j, j = 0,1,\ldots,k+2$ and we obtained the continuous formulation of the form,

$$y(x) = \left[\frac{-2(x-x_n)^6+24h(x-x_n)^5-105h^2(x-x_n)^4+200h^3(x-x_n)^3-144h^4(x-x_n)^2+2'}{27h^6}\right.$$
$$\left[\frac{4(x-x_n)^6-51h(x-x_n)^5+2404h^2(x-x_n)^4-495h^3(x-x_n)^3+378h^4(x-x_n)^2}{76h^6}\right]y_{n+1} +$$
$$\left[\frac{44(x-x_n)^6-447h(x-x_n)^5+1500h^2(x-x_n)^4-1835h^3(x-x_n)^3+738h^4(x-x_n)^2}{2052h^6}\right]y_{n+3} +$$
$$\left[\frac{-35(x-x_n)^6+432h(x-x_n)^5-1986h^2(x-x_n)^4+4184h^3(x-x_n)^3-3963h^4(x-x_n)^2+228h}{1368h^5}\right.$$
$$+\left[\frac{-17(x-x_n)^6+193h(x-x_n)^5-773h^2(x-x_n)^4+1263h^3(x-x_n)^3-666h^4(x-x_n)^2}{228h^5}\right]f_{n+1} +$$
$$\left[\frac{-13(x-x_n)^6+123h(x-x_n)^5-381h^2(x-x_n)^4+445h^3(x-x_n)^3-174h^4(x-x_n)^2}{184h^5}\right]f_{n+3} +$$
$$\left[\frac{(x-x_n)^6-8h(x-x_n)^5+22h^2(x-x_n)^4-24h^3(x-x_n)^3+9h^4(x-x_n)^2}{456h^5}\right]f_{n+4} \qquad (2.1)$$

From the continuous formulation of equation (2.1), when evaluated at $x = x_{n+j,}\ j = 4, 2, \frac{3}{2}$ and the first derivative of equation (2.1) at $x = x_{n+j}\ j = 1, \frac{3}{2}$ gives the following discrete schemes to form our hybrid block scheme of Quade's type

$$y_{n+4} - \frac{8}{19}(y_{n+3} - y_{n+1}) - y_n = \frac{6h}{19}[f_{n+4} + 4f_{n+3} + 4f_{n+1} + f_n]$$
$$y_{n+2} - \frac{11}{27}y_n - \frac{4}{19}y_{n+1} - \frac{196}{513}y_{n+3} = \frac{h}{342}[41f_n + 240f_{n+1} - 64f_{n+3} + 3f_{n+4}]$$
$$y_{n+\frac{3}{2}} - \frac{199}{2432}y_{n+3} - \frac{1701}{2432}y_{n+1} - \frac{7}{32}y_n = \frac{h}{9728}[597f_n + 5238f_{n+1} - 462f_{n+3} + 275f_{n+4}]$$
$$y_{n+1} - y_{n+3} = \frac{h}{90}[f_n - 34f_{n+1} - 114f_{n+2} - 34f_{n+3} + f_{n+4}]$$
$$91y_{n+3} - 243y_{n+1} + 152y_n = \frac{h}{10}[-445f_n - 2160f_{n+1} + 2432f_{n+\frac{3}{2}} + 500f_{n+3} - 27f_{n+4}] \qquad (2.2)$$

Equation (2.2) is of 0rder $[6, 6, 6, 6, 6]^T$ with error constants:

$$[-\frac{6}{665}, \frac{11}{5985}, \frac{963}{1361920}, -\frac{5}{42}, \frac{351}{56}]^T$$

Solving the block implicit hybrid scheme (2.2) simultaneously, we obtained the following block scheme:

$$y_{n+1} = y_n + \frac{h}{360}\left\{103f_n + 593f_{n+1} - 144f_{n+\frac{3}{2}} + 273f_{n+2} - 37f_{n+3} + 4f_{n+4}\right\}$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{2560}\left\{727f_n + 4752f_{n+1} - 3200f_{n+\frac{3}{2}} + 1782f_{n+2} - 248f_{n+3} + 27f_{n+4}\right\}$$

$$y_{n+2} = y_n + \frac{h}{270}\left\{77f_n + 492f_{n+1} - 256f_{n+\frac{3}{2}} + 252f_{n+2} - 28f_{n+3} + 3f_{n+4}\right\}$$

$$y_{n+3} = y_n + \frac{h}{40}\left\{11 f_n + 81 f_{n+1} - 64 f_{n+\frac{3}{2}} + 81 f_{n+2} - 11 f_{n+3} + 0 f_{n+4}\right\}$$

$$y_{n+4} = y_n + \frac{h}{45}\left\{14 f_n + 64 f_{n+1} + 0 f_{n+\frac{3}{2}} + 24 f_{n+2} + 64 f_{n+3} + 14 f_{n+4}\right\} \quad \ldots\ldots\ldots(2.3)$$

3. REFORMULATION OF THE QTM

Reformulating the block hybrid method (2.3) in the general linear method for first and second derivatives (see Butcher (2003), Butcher (2004) and Chollom and Jackweiz (2003) and the coefficients as characterized by the Butcher array (1.5) and (1.6), we obtained, respectively,

$$\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{103}{360} & \frac{593}{360} & \frac{-8}{5} & \frac{91}{120} & \frac{-37}{360} & \frac{1}{90} \\
\frac{3}{2} & \frac{727}{2560} & \frac{297}{160} & \frac{-5}{4} & \frac{891}{1280} & \frac{-31}{320} & \frac{27}{2560} \\
2 & \frac{77}{270} & \frac{82}{45} & \frac{-128}{135} & \frac{14}{15} & \frac{-14}{135} & \frac{1}{90} \\
3 & \frac{11}{40} & \frac{81}{40} & \frac{-8}{5} & \frac{81}{40} & \frac{11}{40} & 0 \\
4 & \frac{14}{45} & \frac{64}{45} & 0 & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \\
\hline
 & \frac{14}{45} & \frac{64}{45} & 0 & \frac{8}{15} & \frac{64}{45} & \frac{14}{45}
\end{array} \quad \ldots(2.4)$$

$$\begin{array}{c|cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{103}{360} & \frac{593}{360} & \frac{-8}{5} & \frac{91}{120} & \frac{-37}{360} & \frac{1}{90} & \frac{5401}{25920} & \frac{403}{432} & \frac{-482}{405} & \frac{923}{1440} & \frac{-683}{6480} & \frac{23}{1728} \\
\frac{3}{2} & \frac{727}{2560} & \frac{297}{160} & \frac{-5}{4} & \frac{891}{1280} & \frac{-31}{320} & \frac{27}{2560} & \frac{3597}{10240} & \frac{4671}{2560} & \frac{-153}{80} & \frac{5103}{5120} & \frac{-393}{2560} & \frac{189}{10240} \\
2 & \frac{77}{270} & \frac{82}{45} & \frac{-128}{135} & \frac{14}{15} & \frac{-14}{135} & \frac{1}{90} & \frac{799}{1620} & \frac{371}{135} & \frac{-992}{405} & \frac{25}{18} & \frac{-83}{405} & \frac{13}{540} \\
3 & \frac{11}{40} & \frac{81}{40} & \frac{-8}{5} & \frac{81}{40} & \frac{11}{40} & 0 & \frac{249}{320} & \frac{369}{80} & \frac{-18}{5} & \frac{459}{160} & \frac{-3}{16} & \frac{9}{320} \\
4 & \frac{14}{45} & \frac{64}{45} & 0 & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & \frac{249}{320} & \frac{896}{135} & \frac{-2048}{405} & \frac{208}{45} & \frac{256}{405} & \frac{16}{135} \\
\hline
 & \frac{14}{45} & \frac{64}{45} & 0 & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & \frac{424}{405} & \frac{896}{135} & \frac{-2048}{405} & \frac{208}{45} & \frac{256}{405} & \frac{16}{135}
\end{array} \quad \ldots(2.5)$$

As characterized by the theory of Nystrôm method (see Haire and Wanner (1976) Butcher (2005)).

An s-stage implicit Runge – Kutta method for the direct integration of second order initial value problem (1.2) and (1.3) is determined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j \qquad \ldots(2.6)$$

$$y'_{n+1} = y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad \ldots(2.7)$$

Where for $i = 1, 2$ - - - - s, a_{ij} =A, $\bar{a}_{ij}$ =Ā (see (1.6))

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j, y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j) \qquad \ldots(2.8)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method.

Using equation (2.6-2.8) we obtained a four-step implicit Runge-Kutta collocation family of uniform order six everywhere on the interval of solution Yakub(2007),

$$k_1 = f(x_n, y_n, y'_n) \quad k_2 = f(x_n + h, y_n + hy'_n + h^2(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6),$$
$$y'_n + h(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6))$$

$$k_3 = f(x_n + \frac{3}{2}h, y_n + \frac{3}{2}hy'_n + h^2(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6),$$
$$y'_n + h(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6))$$

$$k_4 = f(x_n + 2h, y_n + 2hy'_n + h^2(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6),$$
$$y'_n + h(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6))$$

$$k_5 = f(x_n + 3h, y_n + 3hy'_n + h^2(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6),$$
$$y'_n + h(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6))$$

$$k_6 = f(x_n + 4h, y_n + 4hy'_n + h^2(\frac{424}{405}k_1 + \frac{896}{135}k_2 - \frac{2048}{405}k_3 + \frac{208}{45}k_4 + \frac{256}{405}k_5 + \frac{16}{135}k_6),$$
$$y'_n + h(\frac{14}{45}k_1 + \frac{64}{45}k_2 + 0k_3 + \frac{8}{15}k_4 + \frac{64}{45}k_5 + \frac{14}{45}k_6))$$

$$y_{n+1} = y_n + hy'_n + h^2(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6),$$
$$y'_{n+1} = y'_n + h(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + h^2(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6),$$
$$y'_{n+\frac{3}{2}} = y'_n + h(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6)$$

$$y_{n+2} = y_n + 2hy'_n + h^2(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6),$$
$$y'_{n+2} = y'_n + h(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6)$$

$$y_{n+3} = y_n + 3hy'_n + h^2\left(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6\right),$$

$$y'_{n+3} = y'_n + h\left(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6\right)$$

$$y_{n+4} = y_n + 4hy'_n + h^2\left(\frac{424}{405}k_1 + \frac{896}{135}k_2 - \frac{2048}{405}k_3 + \frac{208}{45}k_4 + \frac{256}{405}k_5 + \frac{16}{135}k_6\right),$$

$$y'_{n+4} = y'_n + h\left(\frac{14}{45}k_1 + \frac{64}{45}k_2 + 0k_3 + \frac{8}{15}k_4 + \frac{64}{45}k_5 + \frac{14}{45}k_6\right) \quad \ldots(2.9)$$

4. NUMERICAL EXPERIMENT

To study the efficiency of the method, we present some numerical examples widely used by several authors such as Uwanta et al(2011) using exact finite difference scheme, Ogunfiditimi(2005) using iterative Numerov's scheme and Aregbesola(2004) using linearization method, their approximate solutions were compared with the theoretical solution,the absolute errors (i.e. absolute values of the theoretical solution minus approximate solutions) were compared in table1,table 2 and table 3. The method is applied to solve linear and non-linear second order boundary value problems in ordinary differential equations at block of four points directly without reduction to a system of first order.

Problem 4.1 We consider the BVP $y'' - y = e^{2x}$, $y(0) = 0$, $y(1) = \frac{1}{3}(e^2 - e)$, h=0.2, $0 \le x \le 1$

Theoretical Solution: $y(x) = \frac{1}{3}(e^{2x} - e^x)$

See Uwanta et al (2011)

Problem 4.2 We consider the BVP $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = \frac{2}{1+x^2}$, $y(0) = 0$, $y(1) = 2$, h=0.05, $0 \le x \le 1$

Theoretical Solution: $y(x) = x^2 + x$

See Ogunfiditimi(2005)

Problem 4.3 We consider the non-linear BVP $y'' = \frac{1}{2}(1 + x + y)^3$, $y(0) = 0$, $y(1) = 2$, h=0.2, $0 \le x \le 1$

Theoretical Solution: $y(x) = \frac{2}{2-x} - x - 1$

See Aregbesola(2004)

Table 1: Absolute errors of Problem 4.1

x	Uwanta et al (2011)	PRESENT METHOD
0.0	0.0.E+00	0.0E+00
0.2	6.7.E-03	1.5.E-05
0.4	1.2.E-01	3.4E-05
0.6	1.6.E-01	5.7E-05
0.8	1.3E-01	7.0E+05
1.0	0.0E+00	0.0E+00

Table 2: Absolute errors of Problem 4.2

x	Ogunfiditimi (2005)	PRESENT METHOD
0.0	0.0E-00	0.0.E+00
0.05	5.4E-04	4.8E-07
0.1	9.1E-04	1.4E-06
0.15	1.1E-03	2.4E-06
0.2	1.2E-03	3.4E-06
0.25	1.1E-03	4.0E-10
1.0	0.0E+00	0.0E+00

Table 3: Absolute errors of Problem 4.3

x	Aregbesola(2004)	PRESENT METHOD
0.0	0.0E-00	0.0E-00
0.2	3.20E-04	1.0E-04
0.4	3.3E-04	2.2E-04
0.6	3.9E-04	3.6E-04
0.8	5.3E-04	5.2E-04
1.0	0.0E-00	0.0E-00

5. CONCLUSION

Through the approach presented in this paper, we can give the error constants and the continuous form is also available for dense approximation to the solution of a first order, special and general second order (IVPs or BVPs) ordinary differential equations at block of four points. The method requires less work and possesses a gain in efficiency; the method is self starting with no overlapping of solution models.

REFERENCES

[1] Badmus.A.M and Adegboye Z.A. (2010). Comparison of Two New Quade's Type blocks Hybrid Method for Solution of Ordinary Differential Equations, J. of Research in Physical Science, 6, No 1, 71-76.

[2] Butcher J.C and Hojjat G. (2005) Second Derivative Methods with Runge-Kutta Stability, Num alg, 40, 415-429.

[3] Butcher J.C and Wright W.M (2003) The Construction of Practical General Linear Methods, BIT, 43, 695-721.

[4] Butcher J.C Jackiewicz Z (2004) Construction of General Linear Method with Runge-Kutta Stability Propertie, Numerical Algorithms, 36, 53-72.

[5] Chollom, J.P and Jackiewicz Z. (2003) Construction of two step Runge-Kutta (TSRK) methods with large regions of absolute stabilit, J. Campt and Appl maths, 157, 125-137.

[6] Hairer E. and Warner G. (1976). A Theory for Nystrom Method, Numerical Math, 25, 383-400.

[7] Kulikov, G. Yu (2003) Symmetric Runge Kutta method and their Stabilit, Russ J. Numeric Analyze and maths. Modeling, 18, (1) 13-41.

[8] Lambert J D (1973) Computational Methods in Ordinary Differential Equations, John Willey, New York 278p.

[9] Uwanta I.J.,Ajayi O.O.and Talabi S.O(2011) Derivation of Exact Finite Difference Schemes for Ordinary Differential Equations, Abacus, 38, No 2, 25-36.

[10] Yahaya Y.A. and Adegboye Z.A. (2007). A New Quade's Type 4-step block Hybrid Multstep Method for Accurate and Efficient Parallel Solution of Ordinary Differential Equations, Abacus 34,No 2B, 271-278.

[11] Yakub D.G.,Markus S.,Adamu M.Y.and Buba S.S (2007). A New Fomulation for Symmetric Runge Kutta Method Initial Value Problems of Ordinary Differential Equations, Abacus, 34, No 2A, 193-205.

[12] Ogunfiditimi,F. O.(2005) Iterative Numerov's Scheme for the Solution of linear Boundary Value Problems,Abacus, 32, No 2B, 249-252.

[13] Aregbesola,Y.A.S.(2004) Linearization and Numerical Solution of Nonlinear Boundary Value Problems, Abacus, 31, No 2A,108-118.

[14] Yusuph,Y.A. and Onumanyi P (2002) New Multiple Finite Difference Method through Multi –Step Collocation for y"= f (x,y), Abacus, 29; 92-96.

[15] Yahaya Y.A. and Adegboye Z.A. (2011).Reformulation of Quade's Type Four-step block Hybrid Multstep Method into Runge-Kutta Method for Solution of First and Second Order Ordinary Differential Equations. Abacus 38,No 2: 114-124.