International Journal of Science and Technology Volume 3 No. 1, January, 2014

Modification of Simpson's Block Hybrid Multistep Method for General Second Order ODEs

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ABSTRACT

In this paper, we present the Simpson's block method. The process produces Simpson's scheme and some hybrid form which are combined together to form a block method. The method is extended to the case in which the approximate solution to a second order (special or general), as well as first order Initial Value Problems(IVPs) can be calculated using the direct method as those invented by Nyström, its A-stable, Zero-Stable, Consistent and has an implicit structure for efficient implementation. The efficiency of the method is achieved as shown in the table of results.

Keywords: *Simpson's method, Block method, Nyström method,First and Second Order ODEs*

1. INTRODUCTION

Differentials equation play an important role in modeling virtually every physical, technical or biological process from celestial motion to bridge design, to interaction between neurons. As an example, consider the propagation of light and sound in the atmosphere, and of waves in the surface of a pond. All of them may be describe by the same second order differential equations.

Differential equation such as those used to solve real-life problems may not necessarily be directly solvable i.e. do not have closed from solution instead, solutions can be approximated using numerical method (see P. Blanchard(2006)).

Linear multi-step methods constitute a powerful class of numerical procedures for solving order differential equations and Simpson's rule is the most accurate implicit linear multi-step(two-step) method for solving first order differential equations and can be derived through polynomial interpolation(see Lambert J D (1973)).

In this work, we will reformulate the Simpson's block method into Runge-Kutta method for the solution of initial value problems of the form

$$y' = f(x, y) \qquad y(x_o) = y \qquad (1.1)$$

$$y'' = f(x, y) \qquad y(x_o) = y \qquad y'(x_o) = \beta \qquad (1.2).$$

$$y'' = f(x, y, y') \qquad y(x_o) = y \qquad y'(x_o) = \beta \qquad (1.3)$$

A number of numerical methods for these classes of problems have been extensively developed. On the contrary the problem of the form (1.3) is not commonly discussed in literature.

We consider the numerical solution of the IVP for which the calculation of the second derivative cost little more than first derivative alone. There are several interrelated aims in the search for such method, such as high order, low error constants, satisfactory stability property such as A-stability , low implementation costs and self starting.

We particularly wish to emphasize the combination of a multi-step structure with the use of off-grid points, we seek a method that are multistage and multi-value because it will be convenient to extend the general linear method formulation to the high order Runge – Kutta case (Butcher (2003)) by considering a polynomial.

$$y(x) = \sum_{j=1}^{i-1} \phi_j(x) y_{n+j} + h \sum_{j=1}^{i-1} \varphi_j(x) f(\bar{x}_j, y(\bar{x}_j)) \qquad \ldots\ldots\ldots (1.4)$$

where t denotes the number of interpolation point x_{n+j}, j =0,1--- t-1; and m denotes the distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$ j = 0,1---m-1 chosen from the given step $[x_n, x_{n+k}]$. Here y and f are smooth real N-dimensional vector functions. The numerical constant coefficients ϕ_j ,(j = 0,1---t-1) and $h\ \phi_j$, (j = 0,1---m-1) of (1.4) are to be determined since they are selected so that accurate approximations of well behaved problems step size can be a constant or change in the numerical integration process.

The work is organized as follows, in section 2 we will show how the Butcher's Runge-Kutta methods for the first order differential equations tableau are modified to include second derivatives (that is Runge-Kutta Nyström method), in section 3 we show how the Simpson's block method was constructed, this leads to section 4, where the modification is discussed, Finally, some numerical experiments are presented in section 5.

2. BUTCHER'S RUNGE-KUTTA METHODS FOR THE FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS

For the first order order initial value problem (1.1), Butcher J.C and Wright W.M (2003) defined an s-stage implicit Runge-Kutta method in the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} w_i k_i \qquad \dots (1.5)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + h\sum_{j=1}^{s} a_{ij} k_j) \qquad \dots\dots\dots (1.6)$$

The real parameters α_j, k_i, a_{ij} define the method. This method is characterized by Butcher array as follows

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W \end{array} \qquad \dots\dots\dots (1.7)$$

Butcher (2005) defined an s-stage implicit Runge–Kutta method for the direct integration of second order initial value problem (1.2) and (1.3) is determined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2\sum_{j=1}^{i-1} a_{ij} k_j \qquad \dots(1.8)$$

$$y'_{n+1} = y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad \dots(1.9)$$

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + h^2\sum_{j=1}^{i-1} a_{ij} k_j, y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j) \qquad \dots(2.0)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method. In Butcher – array form we have

$$\begin{array}{c|c|c} \alpha & A & \bar{A} \\ \hline & b^T & \bar{b}^T \end{array} \qquad A = a_{ij} = \beta \qquad \bar{A} = \bar{a}_{ij} = \beta^2$$

$$\beta = \beta e \qquad b = w \qquad \bar{b}^T = w^T\beta \qquad \dots\dots\dots(2.1)$$

Where for i = 1, 2 - - - - s.

Where A denotes the (t+m)*(t+m) real matrix and b and α are real vectors of dimension (t+m) and $\alpha_i \in [0, k]$, ί = 1, 2 --- t+m-1. According to Kulikov (2003), if the matrix A in the Butcher's array is a lower triangular matrix with zero main diagonal, then the method is called explicit.

3. CONSTRUCTION OF THE SIMPSON'S BLOCK METHOD

Consider an approximate solution to (1.1-1.3) in the form of power series

$$y(x) = \sum_{j=o}^{t+m-1} a_j x^j$$

$$a \in R, j = o(1)t + m - 1, Y \in C^m(a,b) \subset P(x) \qquad \dots(2.2)$$

$$y'(x) = \sum_{j=o}^{t+m-1} j a_j x^{j-1} \qquad \dots(2.3)$$

Where a_j 's are the parameters to be determined, t and m are points of interpolation and collocation. To form our matrix D we collocate (2.3) at x_{n+j}, $j = 0,1,2$ and interpolate (2.2)

x_{n+j}, $j=\frac{3}{2},2$. Specifically, k=2, t=2 and m=k+1 yield the following system of equations:

$$\sum_{j=o}^{t+m-1} a_j x^j = y_{n+j} \qquad j=\frac{3}{2},2$$

… (2.4)

$$\sum_{j=o}^{t+m-1} j a_j x^{j-1} = f_{n+j} \qquad j=0,1,2$$

… (2.5)

Following the multi-step collocation of Onumanyi et al (2002), we invert once the matrix D which is of dimension (t+m)*(t+m). The proposed continuous formulation takes the form:

$$y(x)=\alpha_{\frac{3}{2}}(x)y_{\frac{3}{2}}+\alpha_2(x)y_{n+2}+h\{\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}\}$$

When using Maple mathematical software to invert (2.4) and (2.5), obtaining values for $a_j,\ j=0,1,\ldots,k+2$ and we obtained the continuous formulation of the form,

$$y(x)=\frac{\left[64h^2(x-x_n)^2-64h(x-x_n)^3+16(x-x_n)^4\right]}{9h^4}y_{n+\frac{3}{2}}$$

$$+\frac{\left[9h^4-64h^2(x-x_n)^2-64h(x-x_n)^3+16h(x-x_n)^4\right]}{9h^4}y_{n+2}$$

$$+\frac{\left[-36h^4+108h^3(x-x_n)-113h^2(x-x_n)^2+50h(x-x_n)^3-8(x-x_n)^4\right]}{108h^3}f_n$$

$$+\frac{\left[-144h^4+268h^2(x-x_n)^2-238h(x-x_n)^3+64(x-x_n)^4\right]}{108h^3}f_{n+1}$$

$$+\frac{\left[-36h^4+229h^2(x-x_n)^2-238h(x-x_n)^3+64(x-x_n)^4\right]}{108h^3}f$$

…(2.6)

Evaluating equation (2.6) at points x_n and x_{n+1} and its first derivative at point $x_{n+\frac{3}{2}}$ we obtained the Simpson's block hybrid schemes with uniformly accurate order four and error constant $(\frac{-1}{90},\frac{-11}{6480},\frac{-53}{240},)^T$.

$$y_{n+2}-y_n=\frac{h}{3}\{f_n+4f_{n+1}+f_{n+2}\}$$

$$7y_{n+2}-16y_{n+\frac{3}{2}}+9y_{n+1}=\frac{h}{12}\{f_n-32f_{n+1}+19f_{n+2}\}$$

$$y_{n+2}-y_{n+\frac{3}{2}}=\frac{h}{192}\left\{f_n-14f_{n+1}+72f_{n+\frac{3}{2}}+37f_{n+2}\right\}$$

………(2.7)

Solving the block implicit hybrid scheme (2.7) simultaneously, we obtained the following block scheme for solution of problem (1.1):

$$y_{n+1}=y_n+\frac{h}{6}\left\{2f_n+7f_{n+1}-4f_{n+\frac{3}{2}}+f_{n+2}\right\}$$

$$y_{n+\frac{3}{2}}=y_n+\frac{h}{64}\left\{21f_n+90f_{n+1}-24f_{n+\frac{3}{2}}+9f_{n+2}\right\}$$

$$y_{n+2}=y_n+\frac{h}{3}\{f_n+4f_{n+1}+f_{n+2}\}$$

………(2.8)

4. MODIFICATION OF THE SIMPSON'S BLOCK METHOD

Formulating the block hybrid method (2.8) in the general linear method for first and second derivatives (see Butcher (2003), Butcher (2004) and Chollom and Jackweiz (2003)) and the coefficients as characterized by the Butcher array (1.7) and (2.1), we obtained, respectively,

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{7}{6} & \frac{-2}{3} & \frac{1}{6} \\ \frac{3}{2} & \frac{21}{64} & \frac{45}{32} & \frac{-3}{8} & \frac{9}{64} \\ 2 & \frac{1}{3} & \frac{4}{3} & 0 & \frac{1}{3} \\ \hline & \frac{1}{3} & \frac{4}{3} & 0 & \frac{1}{3} \end{array} \qquad \ldots(2.9)$$

Using (2.1) the general linear method for second derivatives as

$$\begin{array}{c|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{7}{6} & \frac{-2}{3} & \frac{1}{6} & \frac{65}{288} & \frac{31}{48} & \frac{-19}{36} & \frac{5}{32} \\ \frac{3}{2} & \frac{21}{64} & \frac{45}{32} & \frac{-3}{8} & \frac{9}{64} & \frac{201}{512} & \frac{333}{256} & \frac{-51}{64} & \frac{117}{512} \\ 2 & \frac{1}{3} & \frac{4}{3} & 0 & \frac{1}{3} & \frac{5}{9} & 2 & \frac{-8}{9} & \frac{1}{3} \\ \hline & \frac{1}{3} & \frac{4}{3} & 0 & \frac{1}{3} & \frac{5}{9} & 2 & \frac{-8}{9} & \frac{1}{3} \end{array}$$

….(3.0)

As characterized by the theory of Nystrôm method (see Haire and Wanner (1976) and Butcher (2005)).

Using equation (3.0) we obtained a two-step implicit Runge-Kutta collocation family of uniform order four everywhere on the interval of solution of problem(1.2) and (1.3) (Yakub(2007)) and Chollom(2003)) as:

$$y_{n+1} = y_n + hy'_n + h^2\left(\frac{65}{288} f_0 + \frac{31}{48} f_1 - \frac{19}{36} f_{\frac{3}{2}} + \frac{5}{32} f_2\right),$$

$$y'_{n+1} = y'_n + h\left(\frac{1}{3} f_0 + \frac{7}{6} f_1 - \frac{2}{3} f_{\frac{3}{2}} + \frac{1}{6} f_2\right)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2} hy'_n + h^2\left(\frac{201}{512} f_0 + \frac{333}{256} f_1 - \frac{51}{80} f_{\frac{3}{2}} + \frac{117}{512} f_2\right),$$

$$y'_{n+\frac{3}{2}} = y'_n + h\left(\frac{21}{64} f_0 + \frac{45}{32} f_1 - \frac{3}{8} f_{\frac{3}{2}} + \frac{9}{64} f_2\right)$$

$$y_{n+2} = y_n + 2hy'_n + h^2\left(\frac{5}{9} f_0 + 2 f_1 - \frac{8}{9} f_{\frac{3}{2}} + \frac{1}{3} f_2\right),$$

$$y'_{n+2} = y'_n + h\left(\frac{1}{3} f_0 + \frac{4}{3} f_1 + 0 f_{\frac{3}{2}} + \frac{1}{3} f_2\right) \qquad (3.1)$$

5. NUMERICAL EXPERIMENT

To study the efficiency of the method, we present some numerical examples widely used by several authors such as Agam S.A and Badmus A.M (2010), Yusuph Y. and Onumanyi P (2002), and Adegboye Z.A.(2008), their approximate solutions were compared with the theoretical solution. The method is applied to solve first order, special and general second order initial value problems in ordinary differential equations at block of two points directly without reduction to a system of first order.

Problem 4.1 $y'' - xy' + 4y = 0$, $y(0) = 3$, $y'(0) = 0$ h=0.1,

Theoretical Solution: $y(x) = x^4 6x^2 + 3$

Problem 4.2 $y'' = -y$, $y(0) = 1$, $y'(0) = 1$, h=0.1, $0 \le x \le 0.4$

Theoretical Solution: $y(x) = \cos x + \sin x$

Problem 4.3 $y' = -y$, $y(0) = 1$, , h=0.1, Theoretical Solution: $y(x) = e^{-x}$

Table 1: Comparing the Solutions of Problem 4.1

x	Theoretical Solution	PRESENT METHOD
0.1	2.9401	2.9401
0.2	2.7616	2.7616
0.3	2.4681	2.4681
0.4	2.0656	2.0656

Table 2: Absolute errors of Problem 4.2

x	NUMEROV (ONUMANYI(2002)	PRESENT METHOD
0.1	2.E-07	9.E-08
0.2	4.E-07	8.E-08
0.3	6.E-07	1.E-07
0.4	7.E-06	1.E-07

Table 3: Absolute errors of Problem 4.3

x	Adegboye Z.A.(2008)	PRESENT METHOD
0.05	9.10E-08	9.09E-08
0.1	8.53E-08	8.56E-08
0.15	1.52E-07	1.521E-07
0.2	1.40E-07	1.40E-07

6. CONCLUSION

Through the approach presented in this paper, we can give the error constants and the continuous form is also available for dense approximation to the solution of a first order, special and general second order ordinary differential equations at block of two points. The method possesses a gain in efficiency (see table 1) and self starting with no overlapping of solution models.

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