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Derivation of an Implicit 4-Point Block Runge-Kutta for Direct Integration of Third Order IVPs and BVPs in ODEs using the Quade's Type Multistep Method

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ABSTRACT

In this paper, we extend the four-step implicit Runge-Kutta method for direct integration of second order initial value ODEs which is an extension of the Quade's Type Multistep (QTM) method (four step block hybrid to an Implicit 4-Point Block Runge-Kutta for Direct Integration Of Third Order initial value problems (IVPs) and boundary value problems (BVPs) in ODEs via the idea as those invented by Nyström. The theory of Nyström method was adopted in the derivation of the method. The method has an implicit structure for efficient implementation and produces simultaneously approximation of the solution of IVPs and BVPs at a block of four points x_{n+i} (i=1,2,3,4). The proposed method was tested with Numerical experiment to illustrate its efficiency and the method can be extended to solve higher order differential equations.

Keywords: *The Quade's Type Multistep(QTM) Method, Implicit Block Runge-Kutta Method, The theory of Nystrom method, Third Order IVPs And BVPs in ODEs*

1. INTRODUCTION

Although it is possible to integrate a third order ODEs by reducing it to first order system and apply one of the method available for such system it seem more natural to provide commercial method in order to integrate the problem directly. The advantage of these approaches lies in the fact that they are able to exploit special information about ODEs and this result in an increase in efficiency (that is, high accuracy at low cost) For instance, it is well know that Runge-Kutta Nyström method for (1.3) involve a real improvement as compared to standard Runge-Kutta method for a given number of stages [4,p.285].

In this paper,we present a six stage Implicit 4-Point Block Runge-Kutta for Direct Integration Of Third Order initial value problems (IVPs) and boundary value problems (BVPs) ODEs with the following advantage such as high order and stage order, low error constant and low implementation cost.

The Quade's Type Four-Step Block Hybrid Multistep Method (QTM) for first order ODEs of the form

$$y' = f(x,y), \qquad y(x_0) = y_0 \tag{1.1}$$

and is given by

$$y_{n+1} = y_n + \frac{h}{360}\left\{103 f_n + 593 f_{n+1} - 144 f_{n+\frac{3}{2}} + 273 f_{n+2} - 37 f_{n+3} + 4 f_{n+4}\right\}$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{2560}\left\{727 f_n + 4752 f_{n+1} - 3200 f_{n+\frac{3}{2}} + 1782 f_{n+2} - 248 f_{n+3} + 27 f_{n+4}\right\}$$

$$y_{n+2} = y_n + \frac{h}{270}\left\{77 f_n + 492 f_{n+1} - 256 f_{n+\frac{3}{2}} + 252 f_{n+2} - 28 f_{n+3} + 3 f_{n+4}\right\}$$

$$y_{n+3} = y_n + \frac{h}{40}\left\{11 f_n + 81 f_{n+1} - 64 f_{n+\frac{3}{2}} + 81 f_{n+2} - 11 f_{n+3} + 0 f_{n+4}\right\}$$

$$y_{n+4} = y_n + \frac{h}{45}\left\{14 f_n + 64 f_{n+1} + 0 f_{n+\frac{3}{2}} + 24 f_{n+2} + 64 f_{n+3} + 14 f_{n+4}\right\} \qquad (1.2)$$

h is the step-size vector chosen , usually h < 1. [1,2,14]

Which was reformulated by [1] into four-step implicit Runge-Kutta method for the solution of initial value problems of the form

$$y_{n+1} = y_n + \frac{h}{360}\left\{103 f_n + 593 f_{n+1} - 144 f_{n+\frac{3}{2}} + 273 f_{n+2} - 37 f_{n+3} + 4 f_{n+4}\right\}$$

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$$y_{n+3} = y_n + \frac{h}{40}\left\{11 f_n + 81 f_{n+1} - 64 f_{n+\frac{3}{2}} + 81 f_{n+2} - 11 f_{n+3} + 0 f_{n+4}\right\}$$

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h is the step-size vector chosen , usually h < 1. [1,2,14]

Which was reformulated by [1] into four-step implicit Runge-Kutta method for the solution of initial value problems of the form

$$y'' = f(x, y, y') \qquad y(x_0) = y_0\,,\ \ y'(x_0) = y'_0 \qquad (1.3)$$

To have

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f(x_n + h, y_n + hy'_n + h^2\left(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6\right),$$
$$y'_n + h\left(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6\right))$$

$$k_3 = f(x_n + \frac{3}{2}h, y_n + \frac{3}{2}hy'_n + h^2\left(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6\right),$$
$$y'_n + h\left(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6\right))$$

$$k_4 = f(x_n + 2h, y_n + 2hy'_n + h^2\left(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6\right),$$
$$y'_n + h\left(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6\right))$$

$$k_5 = f(x_n + 3h, y_n + 3hy'_n + h^2(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6),$$
$$y'_n + h(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6))$$
$$k_6 = f(x_n + 4h, y_n + 4hy'_n + h^2(\frac{424}{405}k_1 + \frac{896}{135}k_2 - \frac{2048}{405}k_3 + \frac{208}{45}k_4 + \frac{256}{405}k_5 + \frac{16}{135}k_6),$$
$$y'_n + h(\frac{14}{45}k_1 + \frac{64}{45}k_2 + 0k_3 + \frac{8}{15}k_4 + \frac{64}{45}k_5 + \frac{14}{45}k_6))$$
$$y_{n+1} = y_n + hy'_n + h^2(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6),$$
$$y'_{n+1} = y'_n + h(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6)$$
$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + h^2(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6),$$
$$y'_{n+\frac{3}{2}} = y'_n + h(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6)$$
$$y_{n+2} = y_n + 2hy'_n + h^2(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6),$$
$$y'_{n+2} = y'_n + h(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6)$$
$$y_{n+3} = y_n + 3hy'_n + h^2(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6),$$
$$y'_{n+3} = y'_n + h(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6)$$
$$y_{n+4} = y_n + 4hy'_n + h^2(\frac{424}{405}k_1 + \frac{896}{135}k_2 - \frac{2048}{405}k_3 + \frac{208}{45}k_4 + \frac{256}{405}k_5 + \frac{16}{135}k_6),$$
$$y'_{n+4} = y'_n + h(\frac{14}{45}k_1 + \frac{64}{45}k_2 + 0k_3 + \frac{8}{15}k_4 + \frac{64}{45}k_5 + \frac{14}{45}k_6) \quad (1.4)\ [1]$$

For the first order differential equations (1.1) Butcher[3,13] defined an s-stage implicit Runge-Kutta methods in the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} w_i k_i \quad (1.5)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + h\sum_{j=1}^{s} a_{ij}k_j) \quad (1.6)$$

The real parameters α_j, k_i, a_{ij} define the method.

An s-stage implicit Runge-kutta Nyström for direct integration of second order IVP (1.1 and 1.3) is defined in the form

$$y_{n+1} = y_n + \alpha_i hy'_n + h^2\sum_{j=1}^{i-1} a_{ij}k_j \quad (1.7)$$

$$y'_{n+1} = y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij}k_j \quad (1.8)$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + h^2\sum_{j=1}^{i-1} a_{ij}k_j, y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij}k_j) \quad (1.9)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method (see [16])

Based on 1.6-2.0 and Taylor series expansion,

The Taylor's series expansion of $y(x+h)$ is given as

$$y(x+h) = y(x) + \frac{h}{!}y'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots + \frac{h^k}{k!}y^k + 0(h)^{k+1}$$

$0(h)^{k+1}$ is the local errors.

As $k \to \infty$, $0(h)^{k+1} \to 0$,

since $0 < h < 1$. (2.0)

We proposed An s-stage implicit Runge-Kutta for direct integration of third order IVPs and BVPs of the form

$$y''' = f(x, y, y', y'') \quad y(x_o) = y$$
$$y'(x_o) = \beta \quad y''(x_o) = \alpha \qquad (2.2)$$

$$y''' = f(x, y, y', y'') \qquad y(x_o) = y$$
$$y(b) = \beta \qquad y'(b) = \alpha \qquad (2.3)$$

as

$$y_{n+1} = y_n + \alpha_i h y'_n + \frac{(\alpha_i h)^2}{2} y''_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j \qquad (2.4)$$

$$y'_{n+1} = y'_n + \alpha_i h y''_n + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad (2.5)$$

$$y''_{n+1} = y''_n + h \sum_{j=1}^{i-1} \bar{\bar{a}}_{ij} k_j \qquad (2.6)$$

Where for $i = 1, 2$ - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + \frac{(\alpha_i h)^2}{2} y'' + h^3 \sum_{j=1}^{i-1} a_{ij} k_j y'_n + \alpha_i^2 h y''_n + + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} k_j, y''_n + h \sum_{j=1}^{i-1} \bar{\bar{a}}_{ij} k_j) \qquad (2.7)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}, \bar{\bar{a}}_{ij}$ define the method.

The paper is organized as follows, in section 2 we will show how the butcher's Runge-Kutta methods for the first order differential equations tableau are modified to include second (that is Runge-Kutta Nyström method) derivatives and which we extended to third derivatives, that will be used in section 3 to illustrate the main derivation Block Implicit Runge-Kutta Method for direct integration of Special and General third order initial valued ODEs from Block Implicit Runge-Kutta Method for direct integration of Second order initial valued ODEs resulting from the Quade's Type Four-Step Block Hybrid Multistep Method(QTM) for first order ODEs, finally, some numerical experiments are presented in section 4.

2. BUTCHER'S RUNGE-KUTTA METHODS FOR THE FIRST ORDER DIFFERENTIAL EQUATIONS

The method (1.6) in Butcher-array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array} \qquad (2.8)$$

While for the Runge–Kutta method for the numerical integration of the second order initial value problem (1.1and 1.2) the method (1.8) in Butcher – array form.

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

A = $a_{ij} = \beta^2$ Ā = $\bar{a}_{ij} = \beta$

$\beta = \beta e$ $\bar{b} = W$ $b = W^T\beta$ (2.9)

(see [15,16])

In Butcher – array form our proposed method for (2.2 and 2.3) is

$$\begin{array}{c|c|c|c} \alpha & \bar{\bar{A}} & \bar{A} & A \\ \hline & \bar{\bar{b}}^T & \bar{b}^T & b \end{array}$$

A= $a_{ij} = \beta^3$ Ā = $\bar{a}_{ij} = \beta^2$ $\bar{\bar{A}} = \bar{\bar{a}}_{ij} = \beta$

$\beta = \beta e$ $\bar{b} = W$ $\bar{b}^T = W^T\beta$ $\bar{\bar{b}}^T = W^T \beta^2$ (3.0)

3. DERIVATION OF THE PRESENT METHOD

Expressing (1.2) in the form (2.8) we have

0	0	0	0	0	0	0
1	$\frac{103}{360}$	$\frac{593}{360}$	$\frac{-8}{5}$	$\frac{91}{120}$	$\frac{-37}{360}$	$\frac{1}{90}$
$\frac{3}{2}$	$\frac{727}{2560}$	$\frac{297}{160}$	$\frac{-5}{4}$	$\frac{891}{1280}$	$\frac{-31}{320}$	$\frac{27}{2560}$
2	$\frac{77}{270}$	$\frac{82}{45}$	$\frac{-128}{135}$	$\frac{14}{15}$	$\frac{-14}{135}$	$\frac{1}{90}$
3	$\frac{11}{40}$	$\frac{81}{40}$	$\frac{-8}{5}$	$\frac{81}{40}$	$\frac{11}{40}$	0
4	$\frac{14}{45}$	$\frac{64}{45}$	0	$\frac{8}{15}$	$\frac{64}{45}$	$\frac{14}{45}$
	$\frac{14}{45}$	$\frac{64}{45}$	0	$\frac{8}{15}$	$\frac{64}{45}$	$\frac{14}{45}$

(3.1)

Expressing (1.4) in the form (2.9) we have

0	0	0	0	0	0	0	0	0	0	0	0	0
1	$\frac{103}{360}$	$\frac{593}{360}$	$\frac{-8}{5}$	$\frac{91}{120}$	$\frac{-37}{360}$	$\frac{1}{90}$	$\frac{5401}{25920}$	$\frac{403}{432}$	$\frac{-482}{405}$	$\frac{923}{1440}$	$\frac{-683}{6480}$	$\frac{23}{1728}$
$\frac{3}{2}$	$\frac{727}{2560}$	$\frac{297}{160}$	$\frac{-5}{4}$	$\frac{891}{1280}$	$\frac{-31}{320}$	$\frac{27}{2560}$	$\frac{3597}{10240}$	$\frac{4671}{2560}$	$\frac{-153}{80}$	$\frac{5103}{5120}$	$\frac{-393}{2560}$	$\frac{189}{10240}$
2	$\frac{77}{270}$	$\frac{82}{45}$	$\frac{-128}{135}$	$\frac{14}{15}$	$\frac{-14}{135}$	$\frac{1}{90}$	$\frac{799}{1620}$	$\frac{371}{135}$	$\frac{-992}{405}$	$\frac{25}{18}$	$\frac{-83}{405}$	$\frac{13}{540}$
3	$\frac{11}{40}$	$\frac{81}{40}$	$\frac{-8}{5}$	$\frac{81}{40}$	$\frac{11}{40}$	0	$\frac{249}{320}$	$\frac{369}{80}$	$\frac{-18}{5}$	$\frac{459}{160}$	$\frac{-3}{16}$	$\frac{9}{320}$
4	$\frac{14}{45}$	$\frac{64}{45}$	0	$\frac{8}{15}$	$\frac{64}{45}$	$\frac{14}{45}$	$\frac{249}{320}$	$\frac{896}{135}$	$\frac{-2048}{405}$	$\frac{208}{45}$	$\frac{256}{405}$	$\frac{16}{135}$
	$\frac{14}{45}$	$\frac{64}{45}$	0	$\frac{8}{15}$	$\frac{64}{45}$	$\frac{14}{45}$	$\frac{424}{405}$	$\frac{896}{135}$	$\frac{-2048}{405}$	$\frac{208}{45}$	$\frac{256}{405}$	$\frac{16}{135}$

(3.2)

Hence A and b in (3.0) are given by

$$b = \begin{bmatrix} \frac{484}{243} & \frac{4624}{405} & \frac{-11776}{1215} & \frac{968}{135} & \frac{-80}{243} & \frac{44}{405} \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{27023}{311040} & \frac{7801}{25920} & \frac{-1079}{2430} & \frac{4681}{17280} & \frac{-4441}{77760} & \frac{941}{103680} \\ \frac{1857}{8192} & \frac{10107}{10240} & \frac{-393}{80} & \frac{13959}{20480} & \frac{-249}{2048} & \frac{693}{4096} \\ \frac{8513}{19440} & \frac{3451}{1620} & \frac{-2824}{1215} & \frac{275}{216} & \frac{-1027}{4860} & \frac{179}{6480} \\ \frac{1373}{1280} & \frac{1857}{320} & \frac{-53}{10} & \frac{423}{128} & \frac{-139}{320} & \frac{69}{1280} \\ \frac{484}{243} & \frac{4624}{405} & \frac{-11776}{1215} & \frac{968}{135} & \frac{-80}{243} & \frac{44}{405} \end{bmatrix} \quad (3.3)$$

Using equation (3.3) in (2.1-2.4) we obtained an implicit 4-point Block Runge-Kutta family of uniform order four everywhere on the interval of solution (Yakub(2007) and chollom(2003)):

$$k_1 = f(x_n, y_n, y'_n, y''_n)$$

$$k_2 = f(x_n + h, y_n + hy'_n + \frac{h^2}{2}y''_n + h^3(\frac{27023}{311040}k_1 + \frac{7801}{25920}k_2 - \frac{1079}{2430}k_3 + \frac{4681}{17280}k_4 - \frac{4441}{77760}k_5 + \frac{941}{103680}k_6),$$
$$y'_n + hy''_n + h^2(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6),$$
$$y''_n + h(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6))$$

$$k_3 = f(x_n + \frac{3}{2}h, y_n + \frac{3}{2}hy'_n + \frac{(\frac{3}{2}h)^2}{2}y''_n + h^3(\frac{1857}{8192}k_1 + \frac{10107}{10240}k_2 - \frac{393}{80}k_3 + \frac{13959}{20480}k_4 - \frac{249}{2048}k_5 + \frac{693}{4096}k_6,$$
$$y'_n + \frac{3}{2}hy''_n + h^2(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6),$$
$$y''_n + h(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6)$$

$$k_4 = f(x_n + 2h, y_n + 2hy'_n + \frac{(2h)^2}{2}y''_n + h^3(\frac{8513}{19440}k_1 + \frac{3451}{1620}k_2 - \frac{2824}{1215}k_3 + \frac{275}{216}k_4 - \frac{1027}{4860}k_5 + \frac{179}{6480}k_6),$$
$$y'_n + 2hy''_n + h^2(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6),$$
$$y''_n + h(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6)$$

$$k_5 = f(x_n + 3h, y_n + 3hy_n' + \frac{(3h)^2}{2} y_n'' + h^3(\frac{1373}{1280}k_1 + \frac{1857}{320}k_2 - \frac{53}{10}k_3 + \frac{423}{128}k_4 - \frac{139}{320}k_5 + \frac{69}{1280}k_6),$$

$$y_n' + 3hy_n'' + h^2(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6),$$

$$y_n'' + h(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6))$$

$$k_6 = f(x_n + 4h, y_n + 4hy_n' + \frac{(4h)^2}{2} y_n'' + h^3(\frac{484}{243}k_1 + \frac{4624}{405}k_2 - \frac{11776}{1215}k_3 + \frac{968}{135}k_4 - \frac{80}{243}k_5 + \frac{44}{405}k_6),$$

$$y_n' + 4hy_n'' + h^2(\frac{424}{405}k_1 + \frac{896}{135}k_2 - \frac{2048}{405}k_3 + \frac{208}{45}k_4 + \frac{256}{405}k_5 + \frac{16}{135}k_6),$$

$$y_n'' + h(\frac{14}{45}k_1 + \frac{64}{45}k_2 + 0k_3 + \frac{8}{15}k_4 + \frac{64}{45}k_5 + \frac{14}{45}k_6))$$

$$y_{n+4} = y_n + 4hy_n' + \frac{(4h)^2}{2} y_n'' + h^3(\frac{484}{243}k_1 + \frac{4624}{405}k_2 - \frac{11776}{1215}k_3 + \frac{968}{135}k_4 - \frac{80}{243}k_5 + \frac{44}{405}k_6$$

$$y_{n+4}' = y_n' + hy_n'' + h^2(\frac{5401}{25920}k_1 + \frac{403}{432}k_2 - \frac{482}{405}k_3 + \frac{923}{1440}k_4 - \frac{683}{6480}k_5 + \frac{23}{1728}k_6),$$

$$y_{n+4}'' = y_n'' + h(\frac{103}{360}k_1 + \frac{593}{360}k_2 - \frac{8}{5}k_3 + \frac{91}{120}k_4 - \frac{37}{360}k_5 + \frac{1}{90}k_6)$$

$$y_{n+3} = y_n + 3hy_n' + \frac{(3h)^2}{2} y_n'' + h^3(\frac{1373}{1280}k_1 + \frac{1857}{320}k_2 - \frac{53}{10}k_3 + \frac{423}{128}k_4 - \frac{139}{320}k_5 + \frac{69}{1280}k_6,$$

$$y_{n+3}' = y_n' + 3hy_n'' + h^2(\frac{249}{320}k_1 + \frac{369}{80}k_2 - \frac{81}{5}k_3 + \frac{459}{160}k_4 - \frac{3}{16}k_5 + \frac{9}{320}k_6),$$

$$y_{n+3}'' = y_n'' + h(\frac{11}{40}k_1 + \frac{81}{40}k_2 - \frac{8}{5}k_3 + \frac{81}{40}k_4 + \frac{11}{40}k_5 + 0k_6)$$

$$y_{n+2} = y_n + 2hy_n' + \frac{(2h)^2}{2} y_n'' + h^3(\frac{8513}{19440}k_1 + \frac{3451}{1620}k_2 - \frac{2824}{1215}k_3 + \frac{275}{216}k_4 - \frac{1027}{4860}k_5 + \frac{179}{6480}k_6,$$

$$y_{n+2}' = y_n' + 2hy_n'' + h^2(\frac{799}{1620}k_1 + \frac{371}{135}k_2 - \frac{992}{405}k_3 + \frac{25}{18}k_4 - \frac{83}{405}k_5 + \frac{13}{540}k_6),$$

$$y_{n+2}'' = y_n'' + h(\frac{77}{270}k_1 + \frac{82}{45}k_2 - \frac{128}{135}k_3 + \frac{14}{15}k_4 - \frac{14}{135}k_5 + \frac{1}{90}k_6)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy_n' + \frac{(\frac{3}{2}h)^2}{2} y_n'' + h^3(\frac{1857}{8192}k_1 + \frac{10107}{10240}k_2 - \frac{393}{80}k_3 + \frac{13959}{20480}k_4 - \frac{249}{2048}k_5 + \frac{693}{4096}k_6,$$

$$y_{n+\frac{3}{2}}' = y_n' + \frac{3}{2}hy_n'' + h^2(\frac{3597}{10240}k_1 + \frac{4671}{2560}k_2 - \frac{153}{80}k_3 + \frac{5103}{5120}k_4 - \frac{393}{2560}k_5 + \frac{189}{10240}k_6),$$

$$y_{n+\frac{3}{2}}' = y_n'' + h(\frac{727}{2560}k_1 + \frac{297}{160}k_2 - \frac{5}{4}k_3 + \frac{891}{1280}k_4 - \frac{31}{320}k_5 + \frac{27}{2560}k_6)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 (\frac{27023}{311040} k_1 + \frac{7801}{25920} k_2 - \frac{1079}{2430} k_3 + \frac{4681}{17280} k_4 - \frac{4441}{77760} k_5 + \frac{941}{103680} k_6,$$

$$y'_{n+1} = y'_n + hy''_n + h^2 (\frac{5401}{25920} k_1 + \frac{403}{432} k_2 - \frac{482}{405} k_3 + \frac{923}{1440} k_4 - \frac{683}{6480} k_5 + \frac{23}{1728} k_6),$$

$$y''_{n+1} = y''_n + h(\frac{103}{360} k_1 + \frac{593}{360} k_2 - \frac{8}{5} k_3 + \frac{91}{120} k_4 - \frac{37}{360} k_5 + \frac{1}{90} k_6)$$

4. NUMERICAL STUDY

Momoniat and Mahomed [27] reduced the third-order ODE $y''' = y - k$ by successive reduction of order to a second-order ODE and then to a first-order ODE.,A fourth-order Runge-Kutta method to solve the first-order ODE $y' = f(x; y)$ gives values $yi = y(xi)$ and h is the step length. An initial value $y(0)$ is chosen with successive values determined by Symmetry Reduction and Numerical Solution of a Third Order ODEs. Using (3.4) to solve this problem gives;

Table 1

x	Exact Solution	RK4 Method	Symmetry Reduction[26]	IRK METHOD
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.221211030	1.221210005	1.221210004	1.221210004
0.4	1.488834893	1.488834780	1.488834779	1.4888834777
0.6	1.807361404	1.807361398	1.807361397	1.807361394
0.8	2.179819234	2.179819234	2.179819233	2.179819218
1.0	2.608275822	2.608274868	2.608274867	2.60827484

Table 1 Comparing numerical values of the numerical solution obtained using a fourth-order Runge-Kutta method, Symmetry Reduction of a Third-OrderODE and the Implicit 4-point Runge-kutta method(IRK Method)(3.4) for direct integration of third order ODEs at $x \in [0; 0{:}2; 0{:}4; 0{:}6; 0{:}8; 1{:}0]$ taking $h = 0{:}01$ and $k = 2$ for the initial conditions $y(0) = y'(0) = y''(0) = 1$.

Problem 2 Consider the boundary value problem

$$y''' - xy = (x^3 - 2x^2 - 5x - 3)e^x,$$

$$y(0) = 0, \quad y'(0) = 1, \quad y'(1) = -e, \; 0 \le x \le 1$$

with h=0.1

Theoretical Solution: $y(x) = x(1-x)e^x$

x	EXACT SOLUTION	THE PRESENT METHOD APPR0XIMATE SOLUTION	ABSOLUTE ERROR OF THE PRESENT METHOD	ABSOLUTE ERROR OF THE STANDARD FINITE DIFFERERENCE METHOD
0.1	0.099465383	9.95E-02	3.35743E-07	3.593003E-05
0.2	0.195424441	0.195425831	1.38999E-06	1.462095E-04
0.3	0.28347035	0.283473504	3.15471E-06	3.189386E-04
0.4	0.358037927	0.358043515	5.58797E-06	5.402571E-04
0.5	0.412180318	0.412166173	1.41443E-05	7.940793E-04
0.6	0.437308512	0.437295341	1.31714E-05	1.061802E-03
0.7	0.422888069	0.422876355	1.17139E-05	1.321959E+03
0.8	0.356086549	0.356076752	9.79656E-06	1.549816E-03
0.9	0.22136428	0.22135677	7.5102E-06	1.716880E-03
1	3.0179E-16	-2.69E-05	2.68753E-05	1.790306E+03

Table 2 Comparing Standard Finite Difference Method[21] and the Block Implicit Runge-kutta method(BIRK Method) for direct integration of third order ODEs

5. CONCLUSION

Through the approach presented in this paper, the BIRK method can be extended to solve higher order differential equations. The method requires less work with little cost (when compared with SFD method) and possesses a gain in efficiency with no overlapping of results.

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