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On Categories of Multisets (MUL) and Topological Spaces (TOP)

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ABSTRACT

The paper briefly described the concept of a category and shows that, the theory generalizes much of mathematics in terms of objects and arrows independent of what the objects and arrows represent. We illustrate the concept of an isomorphism in categorical context and shows that a bimorphism is not necessarily an isomorphism in the categories **Mul** and **Top**.

Keywords: *Category, Isomorphism, Multiset Category.*

1. INTRODUCTION

Emmy Noether, one of the MacLane's teachers, on formalizing *abstract processes*, realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Samuel Eilenberg and Saunders MacLane proposed an axiomatic formalization of the relation between structures and the processes preserving them. In 1945, they presented a paper titled *General Theory of Natural Equivalence* introducing Categories, Functors and Natural Transformations as part of their work in topology, especially algebraic topology. Their goal was that of natural transformations for which the notion of functors and that of categories were exploited (see [10], for details).

The theory of categories became an autonomous field of research that has now occupied a central position in most of the branches of mathematics, some areas of theoretical computer science and mathematical physics. The theory basically studies structures in terms of the mappings between them. In fact many branches of modern mathematics could be conveniently described in terms of categories; for example, category of sets, category of relations, category of groups, etc., and most importantly, doing so often reveals deep insights and similarities between seemingly different areas of mathematics (see [1] and [8], for details).

The notion of category generalizes those of a *preorder* and *monoid* and as well provides unification within set theoretical environment, thereby organizing and unifying much of mathematics. By now it has emerged as a powerful language or a conceptual framework providing tools to characterize the universal components of a family of structures of a given kind and their relationships. The study of categories consists in axiomatically comprehending what is characteristically common in various classes of related mathematical structures by way of exploiting *structure-preserving functions* between them.

2. THE CONCEPT OF A CATEGORY

Essentially, a category is an algebraic structure consisting of a collection of *objects*, linked together by a collection of *arrows* (*morphisms*) that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. Objects and arrows are abstract entities of any kind.

Formally, a Category **C** consists of the following

(i) A Class Ob(**C**) of *objects*.

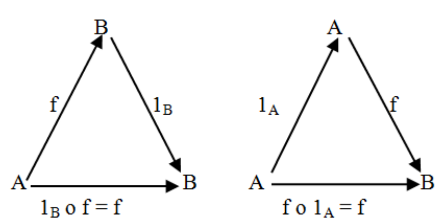
(ii) A class hom(**C**) of *morphisms or maps or arrows*.

(iii) A binary operation called *composition of morphisms*, such that for any three objects A, B and C, we have hom(B, C) x hom(A, B) $\rightarrow$ hom(A, C), (i.e., the composite of f: A $\rightarrow$ B and g: B $\rightarrow$ C is written g o f: A $\rightarrow$ C or gf: A $\rightarrow$ C), governed by two axioms

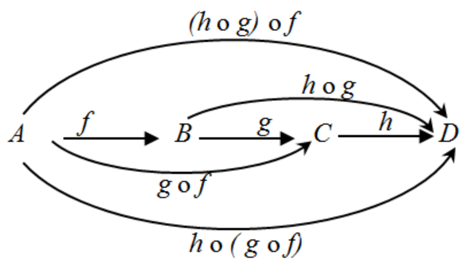
(a) Associativity: if f: A $\rightarrow$ B, g: B $\rightarrow$ C and h: C $\rightarrow$ D then h o (g o f) = (h o g) o f, and

(b) Identity: For every object X, there exists a morphism 1_X: X $\rightarrow$ X called the *identity* morphism for X, such that for every morphism f: A $\rightarrow$ B, we have $1_B \circ f = f = f \circ 1_A$.

Diagramatically, identity and associativity axioms can be viewed as below:



Identity axioms



Associativity axiom

The type of objects depends upon the chosen mathematical structure. For example, in set theory, the objects are sets; in group theory, the objects are groups; and in topology, the objects are topological spaces.

Morphisms are structure-preserving maps. Morphism is an *arrow* linking an object, called the *domain*, to another object, called the *codomain*. A morphism has two parts: the domain (or source) and the codomain (or target). If a morphism f has domain A and codomain B, we write f: A $\rightarrow$ B and say that *f is* a *morphism from A to B*. Thus, morphism is represented by an arrow from its domain to its codomain. The collection *hom-Class* of all morphisms from A to B is denoted *hom(A, B)* or *Hom(A, B)* or $hom_C(A, B)$ or *Mor(A, B)* or ***C****(A, B)*. The notion of morphism connotes differently depending on the type of spaces chosen. In set theory, morphisms are functions; in group theory, they are group homomorphisms; and in topology, they are continuous functions; etc.

Morphism is the central concept in a category and that, a category **C** consists of the collection Mor**C** of the morphisms of **C**. Objects of **C** are associated with identity morphisms 1_A, since 1_A is unique in each set Mor**C**(A, A) and uniquely identifies the object A.

A morphism f: A $\rightarrow$ B in a category **C** is called

(i) a *monomorphism* if it is left cancellable i.e., for every pair of morphisms g,h: C $\rightarrow$ A if f o g = f o h $\Rightarrow$ g = h;

(ii) a *split monomorphism (or section or coretraction)* if it is left invertible i.e., there exists a morphism g: B $\rightarrow$ A such that g o f = 1_A;

(iii) an *epimorphism* if it is right cancellable i.e., for every pair of morphisms g,h: B $\rightarrow$ C, g o f = h o f $\Rightarrow$ g = h;

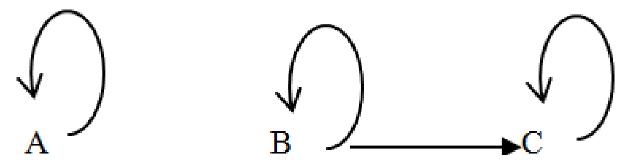
(iv) a *split epimorphism (or retraction)* if it is right invertible i.e., there exists a morphism g: B $\rightarrow$ A such that f o g = 1_B;

(v) a *bimorphism* if it is both a monomorphism and an epimorphism.

Examples of Categories

(i) The simplest nonempty category is the category with one object and exactly one arrow, denoted **1**. The one arrow is actually forced to be the identity map on the one object.

(ii) The category **2** has two objects, their identity arrow and one non-identity arrow as illustrated below.



(iii) Any preordered set (P, $\leq$) forms a *small* category. In this, the objects are the members of P; the morphisms are arrows pointing from x to y when x $\leq$ y, between any two objects there can be at most one morphism. The existence of identity morphism and the composability of the morphisms are guaranteed by the reflexivity and the transitivity of the preorder.

(iv) Any monoid forms a small category with a single object A (A is any fixed set). The morphisms from A to A are precisely the elements of the monoid, the identity morphism of A is the identity of the monoid, and the categorical composition of morphisms corresponds to multiplication of monoid elements.

(v) The class of all sets together with all functions between sets, where composition is the usual function composition, forms a category denoted **Set**. It is the most commonly used category in mathematics.

(vi) The category of finite sets **Fin** has finite sets as objects and functions between finite sets as morphisms.

(vii) The category of relations, denoted **Rel**, has sets as objects and relations as arrows. Composition of R: A $\rightarrow$ B and S: B $\rightarrow$ C is S o R = $\{(a, c)/\exists b\ (a, b)\in R \wedge (b, c)\in S\}$, identities: $1_a = \{(a, a) / a\in A\}$.

(viii) The category of graphs, denoted **Grf**, has graphs as objects and homomorphisms of graphs as morphisms.

Note that a homomorphism ϕ from a graph G to a graph H, denoted ϕ: G $\rightarrow$ H, is a pair of functions $\phi_0: G_0 \rightarrow H_0$ and $\phi_1: G_1 \rightarrow H_1$ with the property that if u: m $\rightarrow$ n is an arrow of G, then $\phi_1(u): \phi_0(m) \rightarrow \phi_0(n)$ in H.

Suppose ϕ: G $\rightarrow$ H and ψ: H $\rightarrow$ K are graph homomorphisms, and suppose that u: m $\rightarrow$ n in G, then by definition $\phi_1(u): \phi_0(m) \rightarrow \phi_0(n)$ in H and so by definition $\psi_1(\phi_1(u)): \psi_0(\phi_0(m)) \rightarrow \psi_0(\phi_0(n))$ in K. Hence ψ o ϕ is a graph homomorphism.

The identity homomorphism id_G is the identity function for both nodes and arrows.

(ix) The category of groups, denoted **Grp**, has groups as objects and group homomorphisms as morphisms.

(x) The category of abelian groups, denoted **AbGrp**, is a category having abelian groups as objects and group homomorphisms as morphisms.

(xi) The category of rings, denoted **Rng**, has rings as objects and ring homomorphisms as morphisms.

(xii) The category of topological spaces, denoted **Top**, is a category with topological spaces as objects and continuous functions as morphisms.

(xiii) Every Directed graph can be made into a category by taking the nodes as objects, and all other paths as morphisms typed with their start and end nodes. Composition is concatenation of paths, and the identities are the empty paths.

The aforesaid examples of categories illustrate how category theory organizes and unifies much of mathematics. It is also clear from various examples of categories given above that a category is mainly characterized by its morphisms and not by its objects.

3. ISOMORPHISM

In general, the word *isomorphic* is used in mathematical context to mean *indistinguishable* in form, translating this in categorical language, we mean, a morphism is an isomorphism if it is invertible. Note that the inverse of an arrow in a category is unique.

Proposition: If f: A $\rightarrow$ B has an inverse, it has only one.

Proof: Let f: A $\rightarrow$ B be a morphism in a category and suppose g: B $\rightarrow$ A and h: B $\rightarrow$ A has the property g o

f = h o f = 1_A and f o g = f o h = 1_B.

We are to show that g = h.

We have, g = g o 1_B = g o (f o h) = (g o f) o h = 1_A o h = h.

Formally, a morphism f: A $\rightarrow$ B in a category **C** is called an *isomorphism* if it is both split monomorphism and split epimorphism i.e., if it is invertible.

Two objects A and B are said to be isomorphic if there exists an isomorphism f: A $\rightarrow$ B. We write A $\cong$ B. As a typical example of isomorphism, any group can be seen as a category with a single object in which every morphism is invertible.

Note that, in categorical terms, a *groupoid* is a category in which every arrow is invertible.

Proposition: Every isomorphism is a bimorphism.

Proof: Let f: A $\rightarrow$ B be an isomorphism i.e., $\exists$ k: B $\rightarrow$ A such that k o f = 1_A, f o k = 1_B and suppose f o g = f o h for g,h: C $\rightarrow$ A, g o f = h o f for g,h: B $\rightarrow$ C, we are to show that g = h in both the cases.

We have, g = 1_A o g = (k o f) o g = k o (f o g) = k o (f o h) = (k o f) o h = 1_A o h = h i.e., g = h.

Also, g = g o 1_B = g o (f o k) = (g o f) o k = (h o f) o k = h o (f o k) = h o 1_B = h i.e., g = h.

Hence f is a bimorphism.

The converse of this proposition need not hold in every category, where the converse holds, we called such a category as *balanced* i.e., a category in which every bimorphism is an isomorphism is said to be *balanced.*

Note that, in the category **Top** of topological spaces and continuous mappings, the isomorphisms are exactly the homeomorphisms. Since continuous bijections need not be homeomorphisms, not all bimorphisms are isomorphisms (see [4], for details).

Moreover, in a poset category, every morphism is a bimorphism since between any two objects there is at most one morphism. However, isomorphisms are only the identity morphisms.

4. MULTISETS

A Multiset (*mset*, for short) is an unordered collection of objects in which, unlike a standard (*Cantorian*) set, duplicates or multiples of objects are admitted. In other words, an mset is a collection in which objects may appear more than once and each individual occurrence of an object is called its *element.* All duplicates of an object in an mset are indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms *object* and *element* does enrich the multiset language. However, use of the term *element* alone may suffice if there does not arise any confusion.

The use of square brackets to represent an mset is quasi-general. Thus, an mset containing one occurrence of *a*, two occurrences of *b*, and three occurrences of *c* is notationally written as [[*a, b, b, c, c, c*]] or [*a*, *b*, *b*, *c*, *c*, *c*] or $[a, b, c]_{1,2,3}$ or [*a*, 2*b*, 3*c*] or [*a*.1, *b*.2, *c*.3] or [1/*a*, 2/*b*, 3/*c*] or $[a^1, b^2, c^3]$ or $[a^1b^2c^3]$. For convenience, the curly brackets are also used in place of the square brackets (Blizard [2] and Singh *et al* [9] are excellent expositions on multisets).

For various application purposes, we may regard a multiset [*a, a, b*] as being really of the form [*a, a′, b*] where *a* and *a′* are different objects of the same *sort* and *b* is of different sort from that of *a* and *a′*. In this regard, when elements of multisets are considered, elements of distinct sorts will generally be denoted by distinct letters and elements of the same sort will be denoted by the same letter with dashes distinguishing different elements of that sort.

Formally a multiset A is a pair $\langle A_0, \rho \rangle$ where A_0 is a set and ρ an equivalence relation on A_0. The set A_0 is called *the field of the multiset*. Elements of A_0 in the same equivalence class is said to be of the same *sort* and elements in different equivalence classes will be said to be of different *sorts*. For example, an mset $[a^2, b, c^3, d]$ will be represented as $[a, a', b, c, c', c'', d]$ where a, a' are of the same sort and c, c', c'' are also of the same sort, while b and d are of two different sorts. In other words, various equivalence classes determine the sorts (see [7], for details).

Multiset Functions

Let $A = (A_0, \rho)$ and $B = (B_0, \sigma)$ be multisets. A multiset function from A to B, written as $f: A \to B$, is a function $f: A_0 \to B_0$ which respects sorts i.e., if $a, a' \in A_0$ and $a\rho a'$, then $f(a)\sigma f(a')$. For example, $f: [a, b, c] \to [d, d', e]$, defined by $f(a) = d'$, $f(b) = e$, $f(c) = d$, and $g: [a, a', b] \to [c, c', d, e, f]$, defined by $g(a) = c$, $g(b) = d$, $g(a') = c'$, are multiset functions, while $h: [a, a', b] \to [c, c', d]$, defined by $h(a) = c'$, $h(a') = d$ and $h(b) = c$, is not an mset function since $a, a' \in A_0$ and $a\rho a'$ but $h(a)\not\sigma h(a')$.

Multisets (considered as *objects*) and multiset functions (considered as *morphisms*) together determine the category of multisets, denoted **Mul.**

A morphism $f: A \to B$ in **Mul,** is said to be an *isomorphism* iff $f: A_0 \to B_0$ is a bijection and also has the property that $a\rho a'$ iff $f(a)\ \sigma\ f(a')$.

For example, $f: [a, b, c] \to [d, e, e']$, defined by $f(a) = d$, $f(b) = e'$, $f(c) = e$, is an mset morphism which is a monomorphism and an epimorphism, and therefore a bimorphism, but not an isomorphism: since f(b) and f(c) are of the same sort but b and c are of different sorts. The multiset morphism $f: [a, a', b] \to [c, c', d]$, defined by $f(a) = c'$, $f(b) = d$, $f(a') = c$, is an isomorphism. Moreover, the morphism $g: [a, b] \to [c, d]$, defined by $g(a) = c$, $g(b) = d$, is also an isomorphism.

Thus, a bimorphism in **Mul**, **Top** is not necessarily an isomorphism and consequently both are not *balanced*, whereas **Set, AbGrp, Rng** and **Grp** are balanced since every bimorphism is an isomorphism.

5. CONCLUSION

We have shown that a category is mainly characterized by its morphisms and not by its objects. Moreover, while some categories are Balanced, **Mul** and **Top** are not balanced.

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