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A Study of Group Theory in the Context of Multiset Theory

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ABSTRACT

Unlike set, a multiset is a collection of objects in which repetition of elements is significant. In this paper, the study of classical group theory in the context of multisets is presented.

Keywords: *set, group, multiset*

1. INTRODUCTION

One of the underlying assumptions of Cantorian set theory dictates that no object shall be allowed to appear more than once. The collection $\{a,b,b,c\}$ consequently becomes a set only after deleting the repeated object to become $\{a,b,c\}$. However, this aspect of Cantorian set theory did not go hand in hand with the requirements of various other sciences in seeking mathematical formulation of some of the challenging real life problems. For example, repeated roots of polynomial equations, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule, H_2O, etc. need to be counted for attaining adequacy and exactness.

Intuitively, multisets are generalization of Sets in which elements can occur more than once. Thus, $[a,b]$ and $[a,a,b]$ are distinct multisets (where square brackets are used to indicate that what is being considered is a multiset and not a classical set). The number of occurrences of an element is called its *multiplicity* and the sum of multiplicities of all elements of a multiset is called its *cardinality*.

Research on the theory of multisets has not yet gain ground as it is still in its infant stages. The research carried out so far shows a strong analogy in the behavior of multisets and sets and it is possible to extend some of the main notions and results of sets to that of multisets. Since the case in consideration has a wider generality (compared to that of sets), the results obtained for multisets are technically more complicated.

This paper is an attempt to explore the theoretical aspects of multisets by studying the notion of a group in multiset context. Section 2 presents some basic definitions and preliminary ideas required for the study. In section 3, we put forward a group definition in multiset context and deduce some corresponding results. We suggest some future research directions in section 4.

2. PRELIMINARIES AND NOTATIONS

Definition 2.1

Formally, a multiset (mset for short) M over a set S is a cardinal-valued function. That is, M on S is a map from S to the set $\mathbb{N}$ of natural numbers denoted $M:S\to\mathbb{N}$. For objects $x\in S$, $M(x)$ denotes the number of times x appears in M also called the multiplicity of x in M. The multiplicity of an element in a mset is axiomatized as unique. i.e

$$\forall x\forall n\forall m(M(x)=n\wedge M(x)=m\to n=m)$$

(see Blizard (1989))

$$M=\{k_1/x_1,k_2/x_2,\dots,k_n/x_n\}=[x_1,x_2,\dots,x_n]_{k_1,k_2,\dots,k_n}$$

also means that M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. i.e $M(x_1)=k_1,M(x_2)=k_2,\dots,M(x_n)=k_n$. We denote the cardinality of a mset M by $|M|$ where $|M|=\sum_{i=1}^{n}k_i$. A mset M is finite if and only if its cardinality is finite.

It follows that $M(x) > 0\ \forall x \in M$. However, $M(x) = 0$ if and only if $x \notin M$. An mset $M \in \mathcal{M}(S)$ such that $M(x) = 0\ \forall x \in S$ is called an empty mset. We denote an empty mset M by ϕ and a class of all finite msets over a set S by $\mathcal{M}(S)$ defined:

$$\mathcal{M}(S) = \{ M \mid M : S \to \mathbb{N} \text{ and } M(x) = 0 \text{ for all but finitely many } x \in M \}.$$

Here, S is called the generating set or ground or base set for $\mathcal{M}(S)$.

Definition 2.2

Two msets $M, N \in \mathcal{M}(S)$ are said to be equal denoted $M = N$ if and only if $M(x) = N(x)$ for all $x \in S$.

Definition 2.3

For any mset $M \in \mathcal{M}(S)$, the predicate $set(M)$ is defined:

$set(M) = \phi \vee \forall x \forall n (M(x) = n \to n = 1)$. Thus, for any set S, $set(S)$.

Definition 2.4

Let $M, N \in \mathcal{M}(S)$. M is a *submultiset* (submset for short) of N, denoted $M \subseteq N$ or $N \supseteq M$ if and only if $M(x) \le N(x)\ \ \forall x \in S$.

Theorem 2.1 (Blizard 1989)

Two msets $M, N \in \mathcal{M}(S)$ are equal if and only if $M \subseteq N$ and $N \subseteq M$.

Definition 2.5

Let $M \in \mathcal{M}(S)$. Then the *root set* M^* of M is defined:

$M^* = \{ x \mid x \in S \wedge M(x) > 0 \}$. In other words, if

$M = [x_1, x_2, ..., x_n]_{k_1, k_2, ..., k_n}\quad k_i > 0,\ i \in [1, n]$,

we have $M^* = [x_1, x_2, ..., x_n]$. Note that $set(M^*) \wedge x \in M \leftrightarrow x \in M^*$ and For any $M \in \mathcal{M}(S)$, $M^* \subseteq M$ (see Blizard 1989).

Theorem 2.2 (Blizard 1989)

For any msets $M, N \in \mathcal{M}(S)$, $M \subseteq N \to M^* \subseteq N^*$

Definition 2.6

A submset M of a mset $N \in \mathcal{M}(S)$ is said to be a *full* submset if and only if $M^* = N^*$

Definition 2.7

Let $M, N \in \mathcal{M}(S)$. The following operations are defined:

(i) *union* $M \bigcup N$ is a mset such that $(M \bigcup N)(x) = \max(M(x), N(x))$
(ii) *intersection* $M \bigcap N$ is a mset such that $(M \bigcap N)(x) = \min(M(x), N(x))$
(iii) *arithmetic multiplication* $M.N$ is a mset such that $(M.N)(x) = M(x)N(x)$
(iv) *multiplication by scalar* kM is a mset such that $(kM)(x) = kM(x)$, $k \in \mathbb{N}$
(v) *raising to arithmetic power* M^k is a mset such that $M^k(x) = (M(x))^k$, $k \in \mathbb{N}$.

Theorem 2.3 (Yager 1986)

Let $M_i \in \mathcal{M}(S)\ \ i = 1, 2, ..., k$.

(i) $$\left(\bigcap_{i=1}^{k} M_i \right)^* = \bigcap_{i=1}^{k} M_i^*$$

(ii) $$\left(\bigcup_{i=1}^{k} M_i \right)^* = \bigcup_{i=1}^{k} M_i^*$$

Proposition 2.4

For any $M \in \mathcal{M}(S)$, $set(M^0) \wedge M^0 = M^*$.
Proof:

(i) For any $x \in M^0$, we have $M^0(x) = (M(x))^0 = 1$ (by definition).
Thus, $set(M^0)$ (by definition).
We show that $M^0 \subseteq M^* \wedge M^* \subseteq M^0$.
Now let $x \in M^0$. Clearly, $M(x) > 0$.

Otherwise if $M(x) = 0$ then $M^0(x) = (M(x))^0 = 0^0 = 1$ (a contradiction).

Thus, $M(x) > 0$. Hence, $x \in M^*$ and $M^0 \subseteq M^*$.

Similarly, let $y \in M^*$. Thus, $M(y) > 0$. Thus, $y \in M$ and $M^0(y) = 1 > 0$.

In particular, $y \in M^0$ and $M^* \subseteq M^0$. Hence, $M^0 = M^*$

Proposition 2.5

For any $M \in \mathcal{M}(S)$, $M^* = (M^k)^* = (kM)^*$ for any $k \in \mathbb{N}$ such that $k \geq 1$

Proof:

We show that

$$M^* \subseteq (M^k)^* \subseteq (kM)^* \text{ and } M^* \supseteq (M^k)^* \supseteq (kM)^*$$

Let $x \in M^*$. We have $M(x) > 0$ (by definition).

In particular, $(M(x))^k = M^k(x) > 0$. Hence, $x \in (M^k)^*$ and $M^* \subseteq (M^k)^*$. But $x \in (M^k)^*$ implies that $M^k(x) = (M(x))^k > 0$. In particular, $M(x) > 0$ and $kM(x) > 0$. Thus, $x \in (kM)^*$.

Hence,

$$M^* \subseteq (M^k)^* \subseteq (kM)^* \qquad \text{(i)}$$

On the other hand, let $y \in (kM)^*$. Then $kM(y) > 0$ (by definition).

Since $k \geq 1$, we have $M(y) > 0$ and $(M(y))^k = M^k(y) > 0$. In particular, $y \in (M^k)^*$ and $(M^k)^* \supseteq (kM)^*$. But $y \in (M^k)^*$ implies that $M^k(y) = (M(y))^k > 0$. In particular, $M(y) > 0$ and $y \in M^*$.

Hence,

$$M^* \supseteq (M^k)^* \supseteq (kM)^* \qquad \text{(ii)}$$

From (i) and (ii) above, it is clear that

$$M^* = (M^k)^* = (kM)^* \text{ for any } k \in \mathbb{N} \text{ such that } k \geq 1$$

Proposition 2.6

Let $M_1, M_2, \ldots, M_r$ be submsets of a mset $M \in \mathcal{M}(S)$ such that $M_1 \cup M_2 \cup \ldots \cup M_r = M$.
Then $\bigcup_{i=1}^{r} M_i^* = M^*$.

Proof:

Since $\bigcup_{i=1}^{r} M_r = M$ (by hypothesis), We have $\left(\bigcup_{i=1}^{r} M_i\right)^* = M^*$.

But $\left(\bigcup_{i=1}^{r} M_i\right)^* = \bigcup_{i=1}^{r} M_i^*$ (from theorem 2.3 (ii)). Thus, $\bigcup_{i=1}^{r} M_i^* = M^*$

Definition(Kuku 1974) 2.8

The pair $(G, \circ)$ where G is a set and $\circ$ a binary operation on G is said to be a group if

(i) $\circ$ is a map $\circ : G \times G \to G$
(ii) $x \circ (y \circ z) = (x \circ y) \circ z \ \forall x, y, z \in G$
(iii) $\forall x \in G, \exists e \in G \ni x \circ e = e \circ x = x$
(iv) $\forall x \in G, \exists y \in G \ni x \circ y = y \circ x = e$ where e satisfies the property in (iii) above.

3. MULTIGROUP

Definition 3.1

Let $M \in \mathrm{M}(S)$. The pair $(M, \circ)$ is a multigroup (mgroup for short) if and only if $(M^*, \circ)$ is a group.

Without loss of generality we denote a mgroup $(M, \circ)$ by M and $x \circ y$ by xy

Definition 3.2

Let $M \in \mathrm{M}(S)$ be a mgroup. A non empty submset N of M is a submgroup if and only if N^* is a subgroup of M^*.

Proposition 3.1

Let $M \in \mathrm{M}(S)$ be a mgroup such that $xy = xz$ for all $x, y, z \in M$. Then $y = z$ and $M(y) = M(z)$ (cancellation law for mgroup)

Proof:

Let $M \in \mathrm{M}(S)$ be a mgroup such that $xy = xz$ for all $x, y, z \in M$. Now $x, y, z \in M \leftrightarrow x, y, z \in M^*$ and M^* is a group (by definition).

Thus, $xy = xz \rightarrow y = z$.

Since $y = z$, we have $M(y) = M(z)$ (see(Blizard (1989)), for unique multiplicity axiom)

Proposition 3.2

Let $M \in \mathrm{M}(S)$ be a mgroup. A non empty submset N of M is a submgroup if and only if for all $x, y \in N$, there exist a positive integer k such that $k / xy^{-1} \in N$

Proof:

Let the non empty submset of M be N

Suppose the submset N is a submgroup, then N^* is a subgroup of M^*. In particular, for any $x, y \in N$, we have $x, y \in N^*$, and $xy^{-1} \in N^*$. Thus, $N\left(xy^{-1}\right) > 0$ and there exist a unique positive integer k such that $k / xy^{-1} \in N$.

Conversely, let $x, y \in N$ such that $N(xy^{-1}) = k$ for a unique positive integer k.

Clearly, $x, y \in N^*$ and $xy^{-1} \in N^*$. Recall that $N \subseteq M \rightarrow N^* \subseteq M^*$. Hence N^* is a subgroup of M^*. In particular, N is a submgroup of M

Proposition 3.3

Let $M_1, M_2, \ldots, M_r$ be submgroups of a mgroup $M \in \mathrm{M}(S)$. Then $\bigcap_{i=1}^{r} M_i$ is a submgroup.

Proof:

Since $M_i \subseteq M \;\; \forall i$, we have

$$\min\left(M_1(x), M_2(x), \ldots, M_r(x)\right) \leq M_i(x) \leq M(x) \;\; \forall x$$

In particular, $\bigcap_{i=1}^{r} M_i(x) \leq M(x)$ and $\bigcap_{i=1}^{r} M_i \subseteq M$ (by definition).

Since $M_1, M_2, \ldots, M_r$ are submgroups of a mgroup $M \in \mathrm{M}(S)$, then $M_1^*, M_2^*, \ldots, M_r^*$ are subgroups of the group M^* (by definition).

Hence, $\bigcap_{i=1}^{r} M_i^*$ is a subgroup.

But $\left(\bigcap_{i=1}^{r} M_i\right)^* = \bigcap_{i=1}^{r} M_i^*$ (from theorem 2.3 (i)). Thus, $\bigcap_{i=1}^{r} M_i$ is a submgroup.

Proposition 3.4

Let $M_1, M_2, \ldots, M_r$ be submgroups of a mgroup $M \in \mathrm{M}(S)$. Then

(i) $\prod_{i=1}^{r} M_i$

(ii) M^k for $k \ge 0$

(iii) kM for $k \ge 1$

are mgroups.

Proof:

(i) Since $M \in \mathcal{M}(S)$ is a mgroup, then M^* is a group and $M_1^*, M_2^*, \ldots, M_r^*$ are subgroups (by definition). Let $e \in M^*$ be the identity. Thus $e \in M_i^*$ for all $i \in [1, r]$.

In particular, $e \in M_i$ i.e $M_i(e) > 0$ for all $i \in [1, r]$.

Thus, $\prod_{i=1}^{r} M_i(e) > 0$ and $e \in \prod_{i=1}^{r} M_i$. Hence, $e \in \left(\prod_{i=1}^{r} M_i\right)^*$ (existence of an identity)

Let $x, y \in \left(\prod_{i=1}^{r} M_i\right)^*$. Clearly $x, y \in \prod_{i=1}^{r} M_i$ with $\prod_{i=1}^{r} M_i(x) > 0$ and $\prod_{i=1}^{r} M_i(y) > 0$.

In particular, $x, y \in M_i$ and $x, y \in M_i^*$ for all $i \in [1, r]$ i.e $xy \in M_i^*$ (since M_i^* are subgroups). Hence, $xy \in M_i$ and $xy \in \prod_{i=1}^{r} M_i$. Thus, $xy \in \left(\prod_{i=1}^{r} M_i\right)^*$ (closure property).

Let $x \in \left(\prod_{i=1}^{r} M_i\right)^*$. We show the existence of $y \in \left(\prod_{i=1}^{r} M_i\right)^*$ such that $xy = yx = e$ (existence of inverse elements).

Now $x \in \left(\prod_{i=1}^{r} M_i\right)^* \leftrightarrow x \in \prod_{i=1}^{r} M_i$. In particular, $\prod_{i=1}^{r} M_i(x) > 0$ and $M_i(x) > 0$ for all $i \in [1, r]$. But $M_i(x) > 0 \rightarrow x \in M_i$ and $x \in M_i \leftrightarrow x \in M_i^*$.

Hence, there exist $y \in M_i^*$ such that $xy = yx = e$ (since M_i^* is a subgroup)

In particular, $y \in M_i$ for all $i \in [1, r]$ and $M_i(y) > 0$.

Thus $\prod_{i=1}^{r} M_i(y) > 0$ and $y \in \prod_{i=1}^{r} M_i$. Hence, $y \in \left(\prod_{i=1}^{r} M_i\right)^*$

Let $x, y, z \in \left(\prod_{i=1}^{r} M_i\right)^*$. We show that $x(yz) = (xy)z$ (Associativity).

Now $x, y, z \in \left(\prod_{i=1}^{r} M_i\right)^* \leftrightarrow x, y, z \in \prod_{i=1}^{r} M_i$ .In particular, $\prod_{i=1}^{r} M_i(x), \prod_{i=1}^{r} M_i(y), \prod_{i=1}^{r} M_i(z) > 0$ and $M_i(x), M_i(y), M_i(z) > 0$ for all $i \in [1, r]$.

But $M_i(x), M_i(y), M_i(z) > 0 \rightarrow x, y, z \in M_i$ and $x, y, z \in M_i^*$.

Thus, $x(yz) = (xy)z$ (since M_i^* is a subgroup).

Hence, $\left(\prod_{i=1}^{r} M_i\right)^*$ is a group. In particular, $\prod_{i=1}^{r} M_i$ is a mgroup.

(ii) Given that $M \in \mathcal{M}(S)$ is a mgroup. We have M^* a group (by definition).

But $M^* = \left(M^k\right)^*$ (from propositions 2.4 and 2.5).

Thus, $\left(M^k\right)^*$ is a group. In particular, M^k is a mgroup.

(iii) The result follows from proposition 2.5.

Definition 3.3

A non empty mset $M \in \mathcal{M}(S)$ is called a cyclic mgroup if and only if M^* is a cyclic group. We denote a cyclic mgroup $M \in \mathrm{M}(S)$ generated by the element x by $M = \langle x \rangle_{k_0}^{k_{n-1}} = [e, x, x^2, x^3, ..., x^{n-1}]_{k_0, k_1, k_2, k_3, ..., k_{n-1}}$ where e is the identity element so that $M^* = \{e, x, x^2, x^3, ..., x^{n-1}\}$ is a cyclic group.

Definition 3.4

A non empty submset N of a cyclic mgroup $M \in \mathrm{M}(S)$ is a cyclic submgroup if and only if N^* is a cyclic subgroup of M^*.

Proposition 3.5

Let $M_1, M_2, ..., M_r$ be cyclic submgroups of a cyclic mgroup $M \in \mathrm{M}(S)$. Then $\bigcap_{i=1}^{r} M_i$ is a cyclic submgroup.

Proof:

Since $M_i \subseteq M$ for all $i \in [1, r]$, we have $\bigcap_{i=1}^{r} M_i \subseteq M$ (by definition).

But $\bigcap_{i=1}^{r} M_i \subseteq M \rightarrow \left(\bigcap_{i=1}^{r} M_i\right)^* \subseteq M^*$. Since $M \in \mathrm{M}(S)$ is a cyclic mgroup, we have a cyclic group M^*.

Given that $M_1, M_2, ..., M_r$ are cyclic submgroups of the cyclic mgroup $M \in \mathrm{M}(S)$, we have the cyclic subgroups $M_1^*, M_2^*, ..., M_r^*$ of the cyclic group M^*(by definition).

In particular, $\bigcap_{i=1}^{r} M_i^*$ is a cyclic subgroup of M^*(see Kuku (1974), for details)).

But $\left(\bigcap_{i=1}^{r} M_r\right)^* = \bigcap_{i=1}^{r} M_i^*$ (from theorem 2.3(i)).

Hence, $\left(\bigcap_{i=1}^{r} M_r\right)^*$ is a cyclic subgroup of M^*. In particular, $\bigcap_{i=1}^{r} M_i$ is a cyclic submgroup.

Proposition 3.6

Let $M_1, M_2, ..., M_r$ be cyclic submgroups of a cyclic mgroup $M \in \mathrm{M}(S)$. Then (i) $\prod_{i=1}^{r} M_i$ (ii) M^k for $k \geq 0$ (iii) kM for $k \geq 1$

are cyclic mgroups.

Proof:

(i) Since $M_1, M_2, ..., M_r$ are submgroups of a mgroup $M \in \mathrm{M}(S)$, by proposition 3.4 (i),

$\prod_{i=1}^{r} M_i$ is a mgroup.

We show that $\prod_{i=1}^{r} M_i$ is cyclic.

Let $M = \langle x \rangle_{k_0}^{k_{n-1}} = [e, x, x^2, x^3, ..., x^{n-1}]_{k_0, k_1, k_2, k_3, ..., k_{n-1}}$

and $y \in \left(\prod_{i=1}^{r} M_i\right)^*$.

Clearly $y \in \prod_{i=1}^{r} M_i$ (by definition).

In particular, $y \in M_i$ for all $i \in [1, r]$ and $y \in M_i^*$ Hence, there exist nonnegative integers t such that $y = x^t$ (since M_i^* is a cyclic subgroup of the cyclic group M^* by definition). Thus, $\left(\prod_{i=1}^{r} M_i\right)^* = \langle x \rangle$. In particular, $\prod_{i=1}^{r} M_i$ is cyclic

(ii) For $k \geq 0$, $(M^k)^* = M^*$ (from propositions 2.4 and 2.5).

Since M is a cyclic mgroup, we have M^* a cyclic group and the results follows.

(iii). Since $(kM)^* = M^*$ (from proposition 2.5) and M^* is a cyclic group, the result follows.

We denotes the order of an element x of a mgroup $M \in \mathsf{M}(S)$ by $o(x)$ and that of the mgroup by $|M|$.

Definition 3.5

Let $M \in \mathsf{M}(S)$ be a mgroup and $x \in M$. Then $o(x)$ and $|M|$ are defined:

(i) $o(x) = o(x)^* M(x)$ where $o(x)^*$ is the order of x in the group M^*

(ii) $|M| = \sum_{x \in M^*} M(x)$

4. FUTURE DIRECTIONS

The concepts of lagrange's theorem, homomorphism of groups and symmetric groups remain challenging in the context of multigroup theory.

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