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A 4-Step Implicit Collocation Method for Solution of First and Second Order Odes

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ABSTRACT

The approach of collocation method approximation will be adopted in the derivation of discrete schemes for direct integration second order ordinary differential equation which are combined together to form a block method. The method is extended to the case in which the approximate solution to a second order (special or general), as well as first order Initial Value Problems(IVPs) can be calculated from the same continuous interpolant and is of order five which is A-stable and has an implicit structure for efficient implementation. The method produces simultaneously approximation of the solution of initial value problems at a block of four points x_{n+i} (i=1,2,3,4). Numerical results are given to illustrate the performance method.

Keywords: *Collocation method, Block method , Continuous interpolant, First and Second Order ODEs*

1. INTRODUCTION

In an earlier work (Yahaya and Adegboye (2007)),the authors constructed and implemented a new Quade's type four-step block hybrid multistep method for accurate and efficient parallel solution of a first order Ordinary Differential Equations(ODEs), The result converged better to the exact solution with A-stable region of absolute stability.These same authors in (Yahaya and Adegboye (2008)) derived a family of 4-step block method for special second order ODEs,the efficiency of the method were tested on both linear and non-linear special second order and the stability plot of the block method was made. In Badmus and Adegboye (2010) the authors obtained two different hybrid block schemes of Quade's type from single continuous formulation, and numerical experiments were applied for the purpose of comparison.Also in Yahaya and Adegboye (2011) Reformulated the Quade's Type Four-Step Block Hybrid Multistep Method Into Runge-Kutta Method For Solution Of First And Second Order Odes.

This paper is part of a research effort to reformulate for efficient and accurate use, of linear multi-step methods. Efforts are directed towards generating a 4-step implicit collocation method for the solution of initial value problems of the form

$$y' = f(x,y)$$
$$y(x_o) = y \qquad (1.1)$$

$$y'' = f(x,y) \qquad y(x_o) = y$$
$$y'(x_o) = \beta \qquad (1.2).$$
$$y'' = f(x,y,y') \qquad y(x_o) = y$$
$$y'(x_o) = \beta \qquad (1.3)$$

A number of numerical methods for these classes of problems have been extensively developed. On the contrary a single continuous interpolant method for (1.1,1.2and 1.3) is not commonly discussed. In literature different continuous interpolant methods are used for (1.1) and (1.2,1.3) see Sirisena etal (2004), Awoyemi (2001) and Adee etal (2005),Adesanya etal (2008), Kayode (2005) to mentioned but few.

We consider the numerical solution of the IVP for which the calculation of the second derivative cost little more than first derivative alone. There are several interrelated aims in the search for such method, such as high order, low error constants, satisfactory stability property such as A-stability , low implementation costs and self starting.

We particularly wish to emphasize the combination of a multi-step structure with the use of off-grid points, we seek a method that are multistage and multi-value because it will be convenient to extend the continuous formulation to the lower order case by considering a polynomial.

$$y(x) = \sum_{j=1}^{i-1} \phi_j(x) y_{n+j} + h^2 \sum_{j=1}^{i-1} \varphi_j(x) f(\bar{x}_j, y(\bar{x}_j)) \qquad \text{……… (1.4)}$$

where t denotes the number of interpolation point x_{n+j}, $j = 0,1\text{---} t-1$; and m denotes the distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$ j = 0,1---m-1 chosen from the given step $[x_n, x_{n+k}]$. Here y and f are smooth real N-dimensional vector functions. The numerical constant coefficients ϕ_j ,(j = 0,1---t-1) and $h^2 \phi_j$, (j = 0,1---m-1) of (1.4) are to be determined since they are selected so that accurate approximations of well behaved problems step size can be a constant or change in the numerical integration process.

Definition 1.0: Linear multistep method (LMM)

For first order odes of the form

$y' = f(x, y), \quad x \in [a, b], y(a) = y_0$, the linear multistep method is typically expressed as $\rho(E)y_n = h\delta(E)f_n$ *where* $E^j f_n = f_{n+j}$ and

$$\rho(E) = \sum_{j=0}^{k} \alpha_j E^j \text{ , } \delta(E) = \sum_{j=0}^{k} \beta_j E^j \qquad (1.5)$$

$\alpha_k \neq 0, \quad \alpha^2 + \beta^2 > 0$ are the first and second characteristic polynomials respectively.

These methods however have direct applications to second order differential equation of the form

$$y'' = f(x, y, y')$$
$$y(0) = y_0, \qquad y'(0) = \beta$$

In literature, see Lambert (1991), Kentnagle and Saff (1994).

$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}$, $k \geq 2$ and more compactly

$$\rho(E)y_n = h^2 \delta(E) f_n \qquad (1.6)$$

Definition 1.1

A linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.7)$$

$k \geq 2$ is said to be of order P if $C_0 = C_1 = C_2 = \cdots C_P = C_{P+1} = 0$

but $C_{P+2} \neq 0$ and C_{P+2} is called error constant.

Definition 1.2

An interval (α, β) of the real line is said to be an interval of absolute stability if the method is absolutely stable for $\bar{h} \in (\alpha, \beta)$

Definition 1.3

A numerical method (1.5) is said to be A—stable if its region of absolute stability contains the whole of the left hand half plane $\text{Re}(h\lambda) < 0$

The paper is organized as follows in section 2 we show how the new method was constructed, this leads to section 3, where the Implementation Strategies is discussed, In section 4, some numerical test for the new methods from which the conclusion (section 5) are drawn are presented.

2. CONSTRUCTION OF THE PRESENT METHOD

In this section, we consider the construction of multistep Collocation Method for constant step size h, and given recursive expression for the coefficients. The values of t and m are arbitrary except for Collocation at the mesh points. Let y_{n+j} be approximations to $y(x_{n+j})_j$ where $y_{n+j} = y(x_{n+j})$,j = 0,…k-1. See Awoyemi (1993).Specifically for this method, we allow t =3 , m =4, i.e. interpolation points are $[x_n, x_{n+1}, x_{n+3}]$. While Collocation points are as $[x_n, x_{n+1}, x_{n+3}, x_{n+4}]$ and we use power series approximation as basis function.

The approximate solution to equation (1) is of the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \qquad \textbf{(1.8)}$$

We differentiate equation (2) twice to have:

$$y''(x) = 2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 \qquad \textbf{(1,9)}$$

To obtain our solution matrix equation, our interpolation and collocation conditions along with equations (1.8) and (1.9) yields the following system of equations which when writing in a matrix equation form we have

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{2.0}$$

Solving (2.0) for a_j, j=0(1)6 using matrix inversion techniques and substituting their values into equation(1.8),some algebraic manipulations yields the proposed continuous scheme of the form

$$y(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + \alpha_3(x) y_{n+3} + h^2\{\beta_0(x) f_n + \beta_1(x) f_{n+1} + \beta_3(x) f_{n+3} + \beta_4(x) f_{n+4}\} \tag{2.1}$$

Hence, our continuous scheme is

$$y(x) = \frac{\left[111h^6 - 64h^5(x-x_n) - 120h^3(x-x_n)^3 + 95h^2(x-x_n)^4 - 24h(x-x_n)^5 + 2(x-x_n)^6\right]}{111h^6} y_n$$

$$+ \frac{\left[27h^5(x-x_n) + 120h^3(x-x_n)^3 - 95h^2(x-x_n)^4 + 24h(x-x_n)^5 - 2(x-x_n)^6\right]}{74h^6} y_{n+1}$$

$$+ \frac{\left[47h^5(x-x_n) - 120h^3(x-x_n)^3 - 95h^2(x-x_n)^4 - 24h(x-x_n)^5 + 2(x-x_n)^6\right]}{226h^6} y_{n+3}$$

$$+ \frac{\left[207h^5(x-x_n) + 360h^4(x-x_n)^2 - 190h^3(x-x_n)^3 + 40h^2(x-x_n)^4 - 3h(x-x_n)^5 + (x-x_n)^6\right]}{226h^6} f_n$$

$$+ \frac{\left[-1007h^5(x-x_n) + 21840h^3(x-x_n)^3 - 15070h^2(x-x_n)^4 + 3591h(x-x_n)^5 - 290(x-x_n)^6\right]}{13320h^4} f_{n+1}$$

$$+ \frac{\left[1449h^5(x-x_n) + 3920h^3(x-x_n)^3 - 3350h^2(x-x_n)^4 + 969h(x-x_n)^5 - 90(x-x_n)^6\right]}{13320h^4} f_{n+3}$$

$$+ \frac{\left[27h^5(x-x_n) + 120h^3(x-x_n)^3 - 95h^2(x-x_n)^4 + 24h(x-x_n)^5 - 2(x-x_n)^6\right]}{} f_{n+4}$$

Evaluating the continuous scheme at $x = x_{n+2}$, $x = x_{n+4}$ and its second derivative at $x = x_{n+2}$ yields

$$y_{n+2} + \frac{97}{111} y_n - \frac{67}{37} y_{n+1} - \frac{7}{111} y_{n+3} = \frac{h^2}{36}\{185 f_n + 1962 f_{n+1} + 22 f_{n+3} - 9 f_{n+4}\}$$

$$y_{n+4} - y_n + 2y_{n+1} - 2y_{n+3} = \frac{h^2}{12}\{- f_n - 10 f_{n+1} + 10 f_{n+3} + f_{n+4}\}$$

$$2y_n - 3y_{n+1} + 2y_{n+3} = \frac{h^2}{240}\{37f_n + 432f_{n+1} - 227f_{n+2} + 32f_{n+3} - 3f_{n+4}\}$$

(2.3)

4. IMPLEMENTATION STRATEGIES

To obtain a starter, we differentiate the Continuous Scheme of equation (2.2) once and evaluate at $x = x_0$ when n=0 $y'(x_0) = Z_0$

$$hy'(x_0) + \frac{47}{40}y_3 - \frac{81}{40}y_1 - \frac{128}{40}y_0 = \frac{h^2}{9600}\{2553f_0 + 6714f_1 + 966f_3 - 153f_4\}$$

(2.4)

To solve (1.2) on the sub-interval $[x_0, x_4]$. We combine equations (2.3) and (2.4), Solving the block implicit discerete schemes simultaneously we obtained the following block schemes when $n = 0$ and Taylor series expansion are adopted to evaluate y_{n+j} and y'_{n+j} ($j = 1,2,3,4$), $n = 0$

$$y_{n+1} = y_n + (h)y'_n + \frac{h^2}{1440}\{367f_n + 540f_{n+1} - 282f_{n+2} + 116f_{n+3} - 214f_{n+4}\}$$

$$y_{n+2} = y_n + (2h)y'_n + \frac{h^2}{270}\{159f_n + 432f_{n+1} - 90f_{n+2} + 48f_{n+3} - 9f_{n+4}\}$$

$$y_{n+3} = y_n + (3h)y'_n + \frac{h}{160}\{147f_n + 468f_{n+1} + 54f_{n+2} + 60f_{n+3} - 9f_{n+4}\}$$

$$y_{n+4} = y_n + (4h)y'_n + \frac{h^2}{45}\{56f_n + 192f_{n+1} + 114f_{n+2} + 64f_{n+3} + 0f_{n+4}\} \quad \ldots\ldots\ldots(2.5)$$

Where the form (2.5) has order $(5,5,5,5)^T$ with error constants $\left(\frac{-197}{13320}, \frac{-97}{60}, \frac{113}{555}, \frac{16497}{103600}\right)^T$ and simultaneously provides value for y_1 ,y_2, y_3 and y_4.

To solve (1.1) on the sub-interval $[x_0, x_4]$. We differentiate the Continuous Scheme of equation (2.2) once and evaluate at $x = x_{n+1}$, $x = x_{n+2}$, $x = x_{n+3}$ and $x = x_{n+4}$ we obtained the following block schemes when $n = 0$ with respect to (2.5)

$$Y_1 = y_0 + \frac{h}{720}\{251f_n + 646f_1 - 264f_2 + 106f_3 - 19f_4\}$$

$$Y_2 = y_0 + \frac{h}{90}\{29f_0 + 124f_1 + 24f_2 + 4f_3 - f_4\}$$

$$Y_3 = y_0 + \frac{h}{80}\{27f_0 + 102f_1 + 72f_2 + 42f_3 - 3f_4\}$$

$$Y_4 = y_0 + \frac{h}{15}\{7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\}$$

…(2.6)

Where the form (2.5) has order $(6,6,6,6)^T$ with error constants $\left(\frac{-439}{1935360}, \frac{-17}{120960}, \frac{-19}{71680}, \frac{1}{15120}\right)^T$

and simultaneously provides value for y_1 ,y_2, y_3 and y_4.Where $Y = y$ when solving problem of the form (1.1) and $Y = y'$ when solving problem of the form (1.2) hence to solve (1.3) on the sub-interval $[x_0, x_4]$we combine (2.5) and (2.6).

5. NUMERICAL EXPERIMENTS

To demonstrate the efficiency of the present method, three test examples were solved.

Example (1) [see Yahaya and Badmus (2008)]

To study the efficiency of the method, we present some numerical examples widely used by several authors such as Yahaya and Badmus (2010), Yusuph Y. and Onumanyi P (2002), Yakub etal(2007) and Yahaya and Adegboye (2008), their approximate solutions were compared with the theoretical solution. The method is applied to solve first order, special and general second order initial value problems in ordinary differential equations at block of four points directly without reduction to a system of first order.

Problem 4.1 $y'' - y' = 0$,

$y(0) = 0$, $y'(0) = -1$, h=0.1, $0 \le x \le 0.4$

Theoretical Solution: $y(x) = 1 - e^{x}$

Problem 4.2 $y'' = -y$,

$y(0) = 1$, $y'(0) = 1$, h=0.1, $0 \le x \le 0.4$

Theoretical Solution: $y(x) = \cos x + \sin x$

Problem 4.3 $y' + 20y = 20x^{2} + 2x$,

$y(0) = \frac{1}{3}$, h=0.01,

Theoretical Solution: $y(x) = x^{2} + e^{-20x}$

Problem 4.4 $y'' = 2y^{3}$,

$y(1) = 1$, $y'(1) = -1$, h=0.1,

Theoretical Solution: $y(x) = \frac{1}{x}$

Table 1: Absolute Errors of Problem 4.1

x	LMM(YAHAYA (2010))	Present Method
0.1	9.E-05	8.E+11
0.2	3.E-03	2.E-09
0.3	2.E-03	4.E-10
0.4	5.E-05	5.E-11

Table 2: Absolute Errors of Problem 4.2

x	NUMEROV (ONUMANYI(2002)	Present Method
0.1	2.E-07	1.E-09
0.2	4.E-07	1.E-09
0.3	6.E-07	2.E-10
0.4	7.E-06	1.E-09

Table 3: Absolute Errors of Problem 4.3

x	Yakub(2007)	Present Method
0.05	7.80E-05	1.97E-09
0.1	3.66E-05	1.30E-08
0.15	2.91E-06	6.61E-09
0.2	9.96E-06	3.93E-09

Table 4: Absolute Errors of Problem 4.4

x	Lmm(Yahaya(2008))	Present Method
1.1	1.62E-06	2.30E-07
1.2	4.01E-06	1.00E-07
1.3	6.52E-06	6.60E-07
1.4	8.84E-06	1.50E-07

6. CONCLUSION

Through the approach presented in this paper, we can give the error constants and the continuous form is also available for dense approximation to the solution of a first order, special and general second order ordinary differential equations at block of four points. The method requires less work with very little cost (when compared with classical and improved RK) and possesses a gain in efficiency (when compared with LMM); the method is self starting with no overlapping of solution models.

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