

Critical Analysis of the Foundations of Differential and Integral Calculus

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ABSTRACT

Critical analysis of the generally accepted (standard) foundations of differential and integral calculus is proposed. Methodological basis of the analysis is the unity of formal logic and of rational dialectics. It is shown that the generally accepted foundations are based on the logically and practically erroneous concepts “infinitesimal quantity (uninterruptedly diminishing quantity)”, “derivative”, “derivative as function of variable quantity” and, consequently, represent incorrect basis of mathematics.

Keywords: *Differential and integral calculus, foundations of mathematics, philosophy of mathematics.*

1. INTRODUCTION

As is known, the formalism of differential and integral calculus is widely and successfully used in natural sciences. There are many works (for example, [1-11]) in which current state of this part of mathematics is expounded. However, this does not mean that the problem of logical substantiation of differential and integral calculus is completely solved in 20-21 centuries, and the foundations of differential and integral calculus are not in need of formal-logical analysis now. Recently, necessity of critical analysis of the foundations of differential and integral calculus within the framework of the correct methodological basis – unity of formal logic and of rational dialectics – arises (see, for example, www.gsjournal.net).

Critical analysis is impossible without plausible reasoning. “We fasten our mathematical knowledge with the help of demonstrative reasoning, but we reinforce our assumptions with the help of plausible reasoning. Everything new that we learn about the world is bound up with plausible reasoning. Plausible reasoning is the only type of reasoning that we are interested in everyday affairs. Mathematics in the making resembles any other human knowledge which is in the creation process. You have to guess mathematical theorem before you prove it; you must guess the idea of the proof before you carry out it in detail. The result of the creative work of mathematician is demonstrative reasoning, proof; but proof open up with the help of plausible reasoning, with the help of guess. Demonstrative reasoning and plausible reasoning supplement each other. The solution of the mathematical problem can be also suggested by the Nature; physics provides us with such keys. A mathematical picture would be too narrow without solution with the help of a physical interpretation” [12].

Today, there are no critical mathematical works carried out within the framework of the correct methodological basis –

unity of formal logic and of rational dialectics. The purpose of this work is to propose a critical analysis of the foundations of differential and integral calculus. The critical analysis is based on plausible reasoning within the framework of methodological basis – unity of formal logic and of rational dialectics.

2. PLAUSIBLE FOUNDATIONS OF DIFFERENTIAL AND INTEGRAL CALCULUS

There is a continuous function y of one argument x :

$$y = f(x).$$

2.1 Let the argument x take the increment Δx . New (accrued) value of the argument is $x + \Delta x$. Then the quantity of function y takes increment Δy , and the new (accrued) value of the function will be

$$y + \Delta y = f(x + \Delta x).$$

The increment Δy of the function has form:

$$\Delta y = f(x + \Delta x) - f(x).$$

2.2 If the increment Δx of the argument tends to zero (i.e. $\Delta x \rightarrow 0$), then Δx becomes infinitesimal quantity (i.e. uninterruptedly diminishing quantity) [1-11]. The limit of this tendency is described as follows:

$$\lim_{\Delta x \rightarrow 0} \Delta x = 0.$$

2.3 The concepts of “variable quantity Δx tends to the limit 0”, “variable quantity Δx tending to the limit 0”, and “process of tendency of variable quantity Δx to the limit 0” are not identical to the concept “limit of variable quantity Δx is equal to 0”, i.e. expression $\Delta x \rightarrow 0$ is not identical to the expression $\lim_{\Delta x \rightarrow 0} \Delta x = 0$:

$$(\Delta x \rightarrow 0) \neq \lim_{\Delta x \rightarrow 0} \Delta x.$$

2.4 The concept “ $\Delta x \rightarrow 0$ ” is briefly designated by the symbol dx :

$$dx \equiv (\Delta x \rightarrow 0).$$

The “variable quantity” dx is uninterruptedly diminishing quantity, and it is called the differential of variable quantity x [1-11]. The variable quantity Δx in expression $\Delta x \rightarrow 0$ and the “variable quantity” dx run the set of permissible values not stopping at one of them.

2.5 If $\Delta x \rightarrow 0$, then the increment Δy of the function is infinitely small (infinitesimal): $\Delta y \rightarrow 0$. The limit of this tendency is:

$$\lim_{\Delta x \rightarrow 0} \Delta y = 0.$$

2.6 The concepts “variable quantity Δy tends to the limit 0”, “process of tendency of variable quantity Δy to the limit 0”, and “variable quantity Δy tending to the limit 0” are not identical to the concept of “limit of variable quantity Δy is equal to 0”, i.e. expression $\Delta y \rightarrow 0$ is not identical to the expression $\lim_{\Delta x \rightarrow 0} \Delta y = 0$:

$$(\Delta y \rightarrow 0) \neq \lim_{\Delta x \rightarrow 0} \Delta y.$$

2.7 The concept “ $\Delta y \rightarrow 0$ ” is briefly designated by the symbol dy :

$$dy \equiv (\Delta y \rightarrow 0).$$

The “variable quantity” dy is uninterruptedly diminishing quantity, and it is called the differential of variable quantity y [1-11]. The variable quantity Δy in

the expression $\Delta y \rightarrow 0$ and the “variable quantity” dy run the set of permissible values not stopping at one of them.

2.8 The ratio of the increments and limit of this ratio have the following forms:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

2.9 The ratio of the increments

$$\frac{\Delta y}{\Delta x}$$

before passing to the limit depends on two variable quantities:

- (a) the initial value of the argument x ;
- (b) the quantity of the increment Δx of argument.

But the limit of this ratio under $\Delta x \rightarrow 0$ no longer depend on vanishing Δx because the initial value x of the argument under finding of the mentioned limit is assumed constant (any limit of a variable quantity is constant) [1-11]. Therefore, the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

being constant quantity may be dependent only on the initial value of the argument x . This limit is an expression containing only the letter x , and, consequently, it is a new function y' (or $f'(x)$) of the argument x .

2.10 A new function y' (or $f'(x)$) of the argument x made by mentioned function $y = f(x)$ is called the derivative of this function $y = f(x)$. Emphasizing the fact that this new function made by the function $y = f(x)$ with the help of some process, one designates the derivative with such symbols: y' or $f'(x)$.

2.11 The ratio of the differentials

$$\frac{dy}{dx}$$

has the following sense:

$$\frac{dy}{dx} \equiv \frac{(\Delta y \rightarrow 0)}{(\Delta x \rightarrow 0)}.$$

Obviously,

$$dx = \Delta x, \quad dy = \Delta y$$

$$\frac{dy}{dx} \neq \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

2.12 If relation between $\frac{dy}{dx}$ and $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ has form

$$\frac{dy}{dx} \approx \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}, \text{ i.e. } \frac{dy}{dx} \approx y'$$

or form of strict equality [1-11]

$$\frac{dy}{dx} = y', \text{ i.e. } dy = y' dx,$$

then this relation represents postulate based on intuition and on the following particular assumption:

$$\frac{(\Delta y \rightarrow 0)}{(\Delta x \rightarrow 0)} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

2.13 Stopping of the process $\Delta x \rightarrow 0, \Delta y \rightarrow 0$ and the return from infinitesimal variables (i.e. uninterruptedly diminishing quantity) dx, dy to the finite variable quantities $\Delta x, \Delta y$ not tending to 0 are carried out with operation of integration which is designated by integral \int :

$$dx = d x, \int dx = \int dx, \Delta x = \int dx. \quad x = \int dx + c,$$

where $\Delta x = x - c, c = const$;

$$dy = d y, \int dy = \int dy, \Delta y = \int f'(x) dx, \\ y = \int f'(x) dx + C,$$

where $\Delta y = y - C, C = const$.

above formulae satisfy formal logic law – the law of identity since the left and right sides of formulas have the same sense, belong to the same qualitative determinacy:

$$(\text{infinitely diminishing quantity}) = (\text{infinitely diminishing quantity})$$

and

$$(\text{finite quantity}) = (\text{finite quantity}).$$

3. DISCUSSION

3.1 The main difference between these formulae and the standard (generally accepted) formulae of differential calculus [1-11] is that the standard formulae

not satisfy formal logic law – the law of identity – since the left and right sides of formula do not have the same meaning, do not belong to the same qualitative determinacy. Really, the variable quantities dx and dy are infinitely small quantities (i.e. infinitely diminishing quantities), and variables quantities Δx and Δy are finite quantities (i.e. not infinitely diminishing quantities). From point of view of formal logic (i.e. the law of identity), the relation between the quantities must be the relation of identity:

$$(\text{infinitely diminished quantity}) = (\text{infinitely diminished quantity})$$

and

$$(\text{finite quantity}) = (\text{finite quantity}).$$

In addition, in accordance with the law of contradiction, infinitely small quantities (i.e. infinitely diminishing quantities) and the finite quantities (i.e. not infinitely diminishing quantities) must be connected by the logical relation of negation:

$$(\text{infinitely diminished quantity}) \neq (\text{not infinitely diminished quantity}).$$

But the standard mathematical relations

$$dx = \Delta x, \quad dy = \Delta y$$

contrary to the law of identity and, consequently, represent a logical error.

3.2 In classical mechanics, use of the definition of derivative leads to a logical error. Really, let point M be moved in the positive direction of the axis Ox . Motion is characterized by a change of coordinate $x(t)$ – continuous function of time t (because, by definition, motion is a change in general). If $\lim_{\Delta t \rightarrow 0} \Delta t = 0$, then $\lim_{\Delta t \rightarrow 0} \Delta x = 0$,

i.e., according to practice and formal logic, value of coordinate does not change and, therefore, there is no movement. But, contrary to practice and formal logic, differential calculus and classical mechanics contain the assertion that velocity $\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$ exists without motion.

Then velocity $\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$ is not real (i.e. not physical)

quantity, but fictitious quantity. Consequently, use of non-physical (unreal) quantity (i.e. the first and second derivatives of function) in classical mechanics is a logic error.

3.3 According to formal logic (i.e. the law of identity and the law of contradiction), the following logical relations between the variable quantities must be fulfilled:

(real quantity) = (real quantity),

(unreal quantity) = (unreal quantity),

(real quantity) ≠ (unreal quantity).

But

$$\frac{dy}{dx}$$

in the relation

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

is unreal quantity (mathematical fiction). Consequently, this relation is a logical error.

3.4 Infinitesimal quantity (i.e. uninterruptedly diminishing quantity) cannot take numerical values. Really, if one will substitute, for example, value $\Delta x = 0,1$ in relation

$$dx \equiv (\Delta x \rightarrow 0),$$

then one will obtain meaningless relation:

$$0,1 \rightarrow 0.$$

Variable quantities $dx \equiv (\Delta x \rightarrow 0)$ and $dy \equiv (\Delta y \rightarrow 0)$ tend to zero without taking numerical values. But such behavior of variable quantities contraries to the experience. Consequently, the infinitesimal quantities dx , dy are fictitious quantities, and the concept “infinitesimal quantity (uninterruptedly diminishing quantity)” represents a logical error.

3.5 Infinitesimal quantities (uninterruptedly diminishing quantities), for example, dx and dy , have neither algebraic meaning, nor geometrical meaning because these quantities do not take numerical values and, therefore, have no a quantitative measure. It means that the quantity y' is not a coefficient in the relation

$$dy = y' dx.$$

Moreover, from the formal-logical point of view, expressions of the type $x + dx$ are erroneous because x (i.e. finite quantity) and dx (i.e. infinitely diminished quantity) have different sense, different qualitative determinacy. Also, the derivative $f'(x)$ has no geometric meaning. Indeed, if

$$\frac{\Delta y}{\Delta x} \equiv \text{tg } \alpha$$

(where α is angle between the secant line and abscissa), then the position of the secant line becomes uncertain under $\lim_{\Delta x \rightarrow 0} \Delta x = 0$ and $\lim_{\Delta x \rightarrow 0} \Delta y = 0$ because: (a) the

triangle formed by the straight segments Δx , Δy , and the secant line degenerates (the triangle and the angle α do not exist); (b) a straight line passes through only one point x . But only one point does not determine position of a straight line. Consequently, the quantity

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

does not determine the tangent line.

In accordance with formal logic, the following statement is true: if the variable quantity x takes numerical values $x = x_1$ and $x = x_2$, then

$$x_2 - x_1 \equiv \Delta x_{21}, \quad x_1 + \Delta x_{21} = x_2,$$

where x_2 is the result of the addition operation. The difference between variable quantities x , Δx and their numerical values is expressed with the help of the suffixes. In order to turn from the addition operation on numerical values to the addition operation on variable quantities x and Δx , one should remove the suffixes from numerical expressions. Then one obtains the following relation: $x + \Delta x = x$. This relation is in accord with the formula

$$x = \int dx + c, \quad c = \text{const}$$

only under the condition $c + \Delta x = x$. Under this condition, the expression

$$\Delta y = f(x + \Delta x) - f(x)$$

takes the correct form:

$$\Delta y = f(c + \Delta x) - f(c).$$

It shows that $x \rightarrow c$ under $\Delta x \rightarrow 0$. In this case,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

does not contain variable quantity x and depends only on constant c . This implies that the differential calculus is an incorrect theory because the formula for derivative contains variable quantity x .

4. RESULTS

Thus, the generally accepted (standard) foundations of differential and integral calculus [1-11] are based on the logically and practically erroneous concepts “infinitesimal quantity (uninterruptedly diminishing quantity)”, “derivative”, “derivative as function of variable quantity” and, consequently, represent incorrect basis of mathematics. Indeed, the standard “mathematics is a doctrine in which it is not known that we are talking about and whether it is true that we speak” (Bertrand Russell, British philosopher, logician, essayist, and social critic, best known for his work in mathematical logic and analytic philosophy).

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