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Investigation of Excited Duffing's Oscillator Using Versions of Second Order Runge-Kutta Methods

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ABSTRACT

This investigation derived its strong motivation in the adoption of versions of second-order Runge-Kutta methods where there is presently dearth of relevant literature to re-establish the complicated nature of solution of Duffing oscillator dynamics. The choice of second-order Runge-Kutta methods hinged on its simplest algebraic formulation of relevant coefficients based on Taylor series expansion comparing with its higher order counterpart. Validation of FORTRAN-90 codes of algorithms was achieved by phase plots comparison reference to Dowell (1988) as standard. The nature of simulated solutions were visually determined with scatter plot of phase variables obtained from simultaneous implementation of large number of versions of second-order Runge-Kutta methods in conjunction with the corresponding literature results. Validation results are acceptable to within the accuracy limit of Runge-Kutta methods adopted. The scatter plots on phase plane for cases investigated are well structured and bounded (strange) and compare correspondingly well with literature Poincare sections. This investigation re-establishes the complex nature of solution of Duffings oscillator dynamics. Its established procedures provide an alternative Poincare section method and can be utilised for preliminary verification of system dynamics behaviour subject to confirmation by additional dynamics tests.

Keywords: *Excited Duffing Oscillator, Second order Runge-Kutta, Taylor Series and Poincare Section*

1. INTRODUCTION

Runge kutta technique is unarguably one of the most highly favoured numerical tools in simulating chaos dynamics. The fourth-order Runge-Kutta method has been the preferred numerical integration scheme for solving chaotic problems in non-linear systems (Moorthy et al, 1993). According to the authors' work, the method is considered very accurate but requires very small time-steps and four equation solutions per time-step. These drawbacks hinder the solution of chaotic problems in multi-degree-of-freedom (MDOF) systems. The paper presents the solution of the chaotic problem of impacting single-degree-of-freedom (SDOF) oscillators, using the Newmark method which is computationally efficient and unconditionally stable. The results of their study are compared with those obtained from the fourth-order Runge-Kutta method. It is concluded that the Newmark method with an adequate check on the solution accuracy could give qualitatively the same results as the Runge-Kutta method. The method has the benefit of an extension to MDOF real-life problems of chaos which could be solved using numerical techniques. Chaotic motion of a virtual double pendulum has been studied by Aan et al (2011). Large oscillations of this pendulum were modelled by the system of nonlinear differential equations in the Hamilton form. This system was solved on the worksheet of Computer Package Mathcad numerically, using the Runge-Kutta algorithm of fourth order. The results obtained were demonstrated by some frames of video clips visualizing the chaotic motion of a double physical pendulum. The paper concluded that the results obtained can serve as teaching aids in the process of analytical mechanics for engineering students. A study which utilizes combination of phase plots, time steps distribution and adaptive time steps Runge-Kutta and fifth order algorithms to investigate a harmonically Duffing oscillator has been carried out by Salau and Ajide ($2012_{(a)}$). The object of their study was to visually compare fourth and fifth order Runge-Kutta algorithms performance as tools for seeking the chaotic solutions of a harmonically excited Duffing oscillator. It is deduced that although fifth order algorithms favours higher time steps and as such faster to execute than fourth order for all studied cases, the reliability of results obtained with fourth order worth its higher recorded total computation time steps period.

The objective of Khatami et al (2008) paper is the analytical investigation of the dynamics of vibration of parametrically excited oscillator with strong cubic negative nonlinearity based on Mathieu- Duffing equation. The analytic investigation was conducted by using He's homotopy-perturbation method (HPM).The authors employed Runge-Kutta's algorithm to solve the governing equation via numerical solution. The effects of variation of the parameters on the accuracy of the homotopy- perturbation method were equally studied. A modified Runge–Kutta method with phase-lag of order infinity for the numerical solution of the Schro¨dinger equation and related problems has been developed in Simos and Jesu´ s (2001) paper. The modified method is based on the classical Runge–Kutta method of algebraic order four. The numerical results indicate that the new method developed is more efficient for the numerical solution of the Schro¨dinger equation and related problems than the well known classical Runge–Kutta method of algebraic order four. The behaviour of chaotic dynamical systems is understood by calculating flows in phase space. Stable points can emerge after iterating these flow many times, revealing significant information about the system (Benjamin, 2011). A rigorous integrator has been developed using Taylor models and implemented in the code COSY INFINITY which integrates ODEs and PDEs rigorously. The author was able to illustrate these integrations of various point densities using an eighth-order Runge-Kutta with automatic step size control using reverse communication. Optimum fractal disk dimension algorithms has been used to characterize the evolved strange attractor when adaptive time steps Runge-Kutta fourth and fifth order algorithms are employed to compute simultaneously multiple trajectories of a harmonically excited Duffing oscillator from very close initial conditions (Salau and Ajide, $2012_{(b)}$). The object of the study was to enable visual comparison of the chaos diagrams in the excitation amplitude versus frequency plane. The chaos diagrams obtained by fourth order algorithms is accepted to be more reliable than its fifth order counterpart, its utility as tool for searching possible regions of parameter space where chaotic behaviour/motion exist may require additional dynamic behaviour tests.

Despite the availability of numerous literatures on the choice of Runge kutta as a numerical tool for characterizing nonlinear dynamics, there is presently dearth of relevant literature which re-establishes the complicated nature of solution of Duffing Oscillator dynamics. The goal of this paper is to investigate this dynamics by adopting versions of second-order Runge-Kutta methods. The focus is what geometrical shape will set of results obtained when the same Duffing equation is simulated from the same initial conditions (displacement and velocity) using several versions of second order Runge-Kutta (1,2,3,----n; for very large n). The justification for the choice of second order Runge-Kutta is that it is the simplest method where in the mathematical relationships between coefficients can easily be obtained without much of rigorous derivation.

2. METHODOLOGY

2.1 Harmonically Excited Duffing's Oscillator

The present study investigated normalized governing equation of harmonically excited Duffing system given by equation (1) with reference to Moon (1987), Dowell (1988) and Narayanan and Jayaraman (1989b).

$$\ddot{x}+\gamma\dot{x}-\frac{x}{2}(1-x^2)=P_oSin(\omega t) \qquad (1)$$

In equation (1) x, $\dot{x}$ and $\ddot{x}$ represents respectively displacement, velocity and acceleration of the oscillator about a set datum. The damp coefficient is γ . Amplitude strength of harmonic excitation, excitation frequency and time are respectively P_o, ω and t . According to literature combination of γ = 0.168, P_o = 0.21, and ω = 1 .0 or γ = 0.0168, P_o = 0.09 and ω = 1.0 parameters leads to chaotic behaviour. However Dowell (1988) reported period one, two and four at respective excitation amplitude of 0.177, 0.178 and 0.197 when the damp and excitation frequency are fixed respectively at 0.168 and 1.0. The present investigation utilised family of second order Runge-Kutta method driven at constant time step (0.01) to seek both transient and steady solutions of equation (1) over a total of seventy (70) excitation periods and initial conditions (1, 0). Scatter diagrams were made in each studied case on phase plane with the last computed stable solutions.

2.2 Generalized Runge-Kutta Methods

According to Chapra and Canale (2006) many variations of explicit Runge-Kutta method exist, however all can be cast in the generalized form given by equation (2).

$$x_{i+1}=x_i+\varphi(t_i,x_i,\Delta t)\Delta t \qquad (2)$$

In equation (2) $\varphi(t_i,x_i,\Delta t)$ is called an incremental function and best interpreted as a representative slope of multivariable function $f(x,t)$ over the time interval (Δt). The increment function can be written in general form as in equation (3) for constants $\alpha^{'s}$ and $K^{'s}$.

$$\varphi=\alpha_1K_1+\alpha_2K_2+\cdots+\alpha_nK_n \qquad (3)$$

2.3 Second-Order Runge-Kutta Methods

The second –order version of equation (2) is given by equation (4).

$$x_{i+1}=x_i+(\alpha_1K_1+\alpha_2K_2)\Delta t \qquad (4)$$

In equation (4) $K_1 = f(t_i, x_i)$ and $K_2 = f(t_i + \beta_1 \Delta t, x_i + \eta_{11} K_1 \Delta t)$. The constant coefficients are related as in equations (5) to (7) with details equivalent relationship between equation (4) and Taylor series expansion to the second –order term provided in Chapra and Canale (2006).

$$\alpha_1 + \alpha_2 = 1 \tag{5}$$

$$\alpha_2 \beta_1 = \frac{1}{2} \tag{6}$$

$$\alpha_2 \eta_{11} = \frac{1}{2} \tag{7}$$

The three simultaneous equations (5) to (7) related four unknown constants and suggested non availability of unique set of constants that satisfy these equations. The second –order Runge-Kutta methods is a choice of this investigation because of its simplest nature of the derivation of equations (5) to (7) compare with its higher order counterpart that involve a large amount of tedious, time consuming and error prone algebraic manipulation. More insights are provided by Cartwright and Piro (1992). The present study assumed nine hundred and ninety nine (999) distinct values for α_2 within $0.001 \le \alpha_2 \le 1.000$ using constant step size of 0.001 and solved for other three constants using equations (5) to (7). The combination of corresponding α_1, β_1, η_{11} and the assumed α_2 tagged (V1, V2… V999) were used one after the other to seek the solutions of equation (1). Mathematical arguments by Chapra and Canale (2006) are that each of these versions (V1, V2… V999) would produce same results provided the ordinary differential equations (ODE) have solution that is quadratic, linear or constant and varied results otherwise. It is here that lays the pivot of the present study with one of the objective being to observe any structural pattern of assembly of solution points from versions of second-order Runge-Kutta methods on the phase plane. The last of computed displacement and velocity in the steady solutions realm produced by V1, V2… V999 were used to plot scatter diagrams on the phase plane.

2.4 Parameters of Cases Investigated

A constant time step (Δt = 0.01), initial conditions (1, 0) and corresponding coefficients combination (V1, V2… V999) are common to all investigated cases. The unsteady solutions spanned the first twenty (20) simulation period reference to harmonic excitation.

Case-I: γ = 0.168, P_o = 0.21, and ω = 1.

Case-II: γ = 0.0168, P_o = 0.09, and ω = 1.0

Case-III: γ = 0.168, P_o = 0.197, and ω = 1.

Case-IV: γ = 0.168, P_o = 0.178, and ω = 1.

Case-V: γ = 0.168, P_o = 0.177, and ω = 1.

Case-VI: γ = 0.168, P_o = 0.21, and ω = 1 in addition to table 1. The version (V500) is one of the three most commonly used and preferred.

Table 1: Coefficients combination for selected version of second-order Runge-Kutta methods

Coefficients	Versions of second-order Runge-Kutta methods								
	V100	V200	V300	V400	**V500**	V600	V700	V800	V900
α_2	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900
α_1	0.900	0.800	0.700	0.600	0.500	0.400	0.300	0.200	0.100
β_1	5.000	2.500	1.667	1.250	1.000	0.833	0.714	0.625	0.556
η_{11}	5.000	2.500	1.667	1.250	1.000	0.833	0.714	0.625	0.556

3. RESULTS AND DISCUSSIONS

The second-order Runge-Kutta methods appropriately coded in FORTRAN-90 was used to compute results presented under this section. The phase plots in figures 1 and 2 were used to validate Runge-Kutta algorithms by comparing corresponding figures with its literature counterpart. Figure 1 compare very well its literature counterpart which was computed by higher order Runge-Kutta. The observed significant variation in figure 2 with its literature counterpart can be explained respect to higher order Runge-Kutta difference. Dynamics computations with higher order Runge-Kutta methods are well known to be stable and more reliable than lower order counterpart.

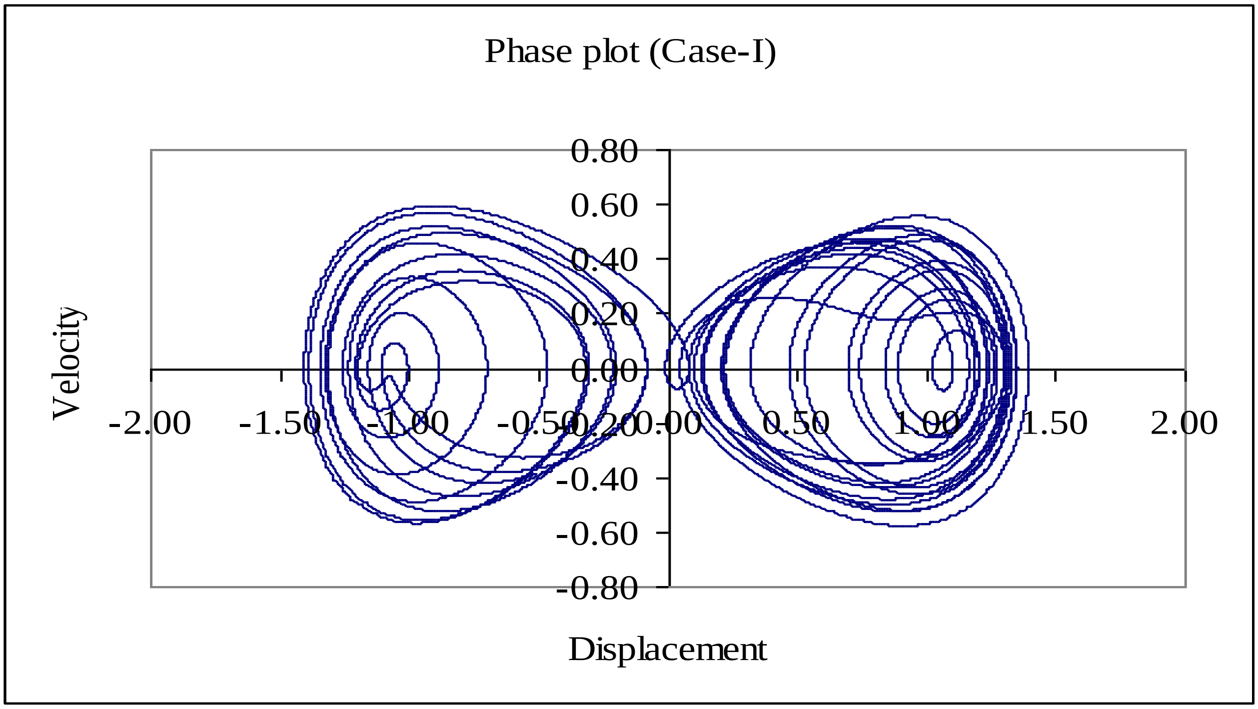

Figure 1: Steady Phase plot (Case-I) computed by V500 over 41st to 70th excitation period

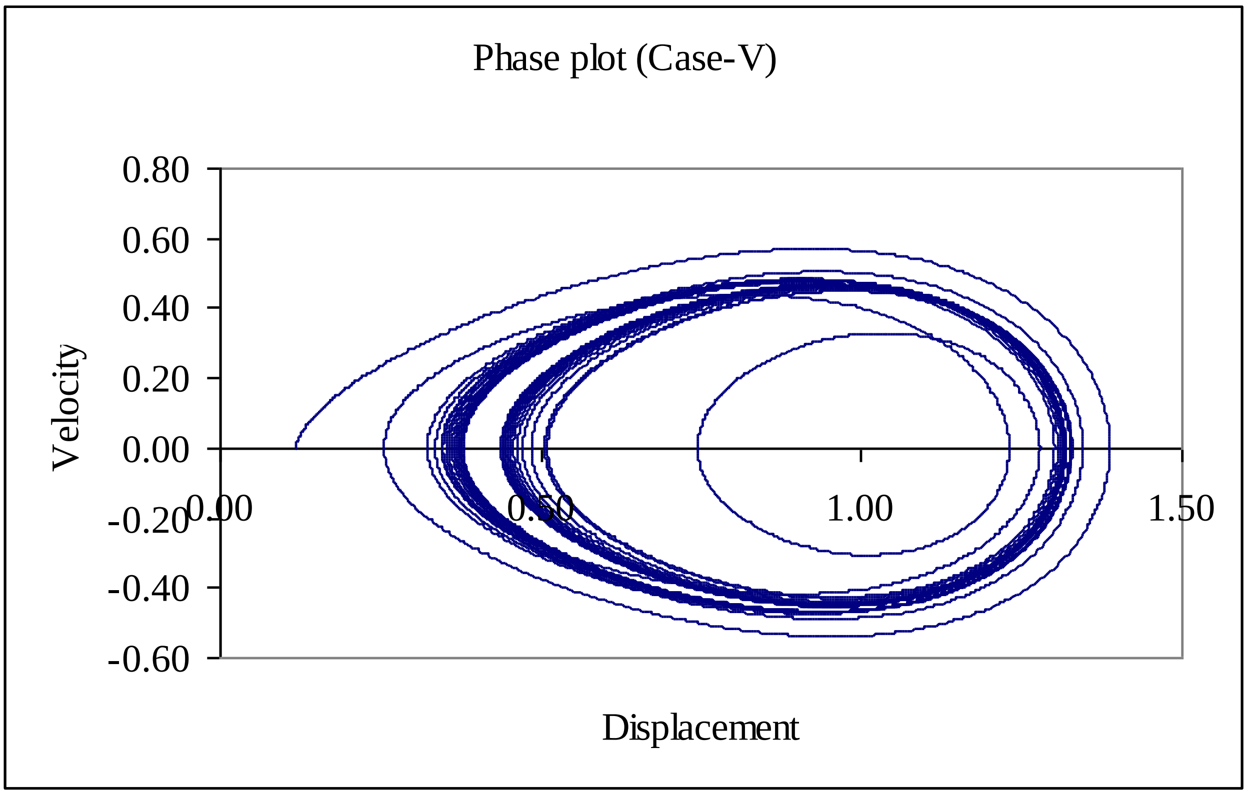

Figure 2: Steady Phase plot (Case-V) computed by V500 over 41st to 70th excitation period

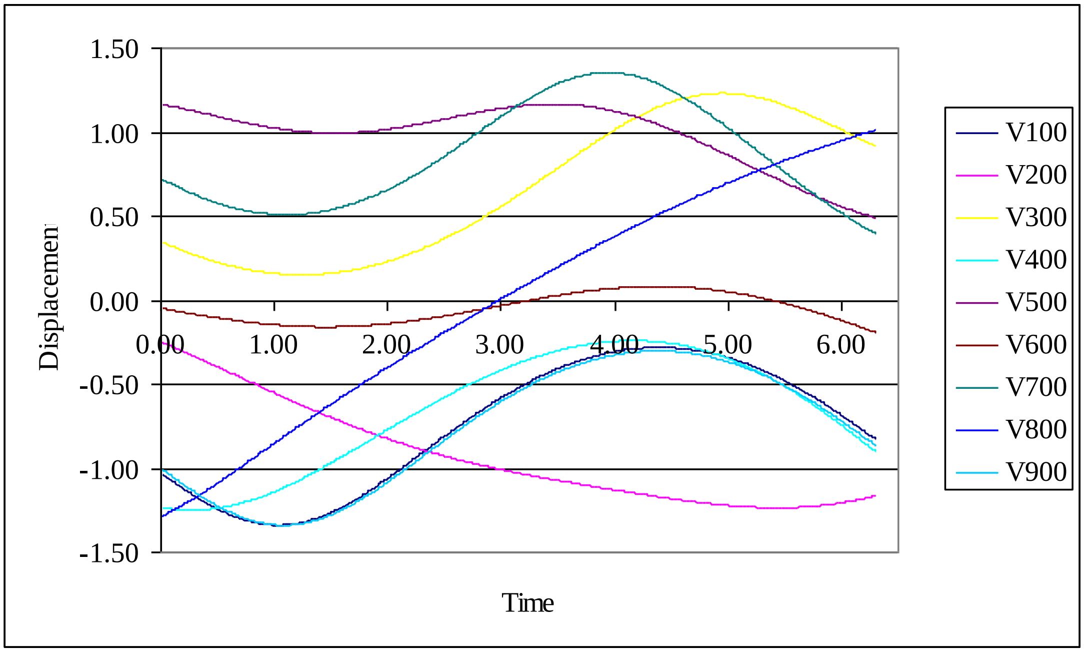

Figure 3: Steady time displacement history (Case-VI) over 70th excitation period

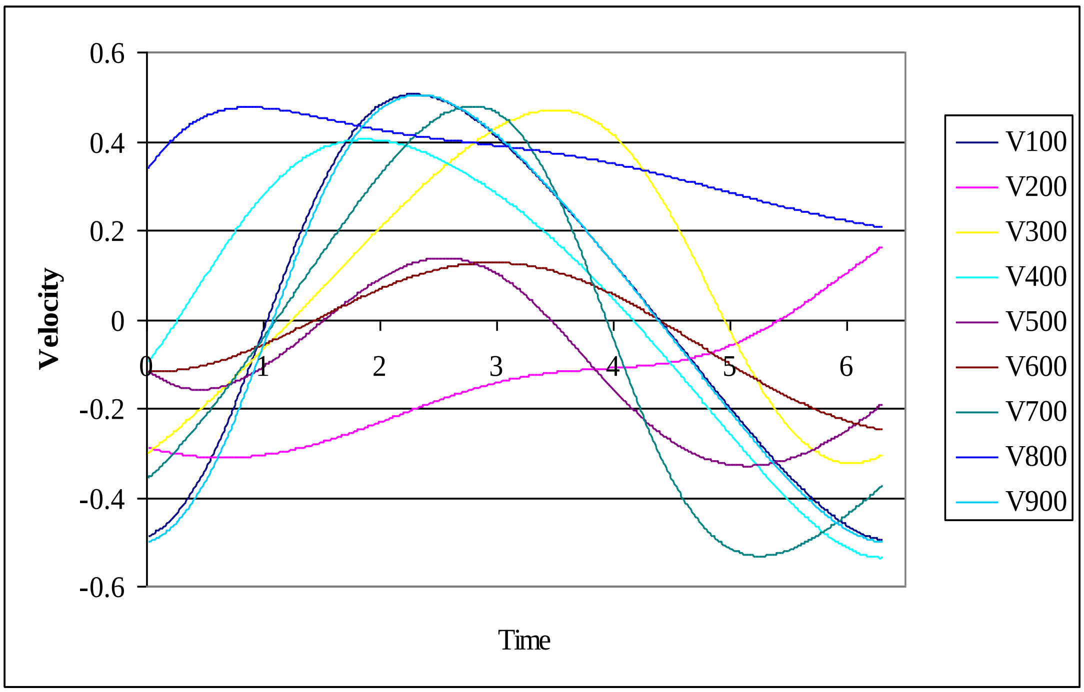

Figure 4: Steady time velocity history (Case-VI) over 70th excitation period.

It is amazing to note that as wildly varied as time history in figures 3 and 4 are the difference among the series computation is the version of the Runge-Kutta methods used. The time displacement history predicted over a whole period of excitation by version V800 was almost linear. The common and preferred version (V500) of second-order Runge-Kutta methods is no exception in the observed wild variations. Furthermore the series in figures 3 and 4 impresses large number of objects swarming at different rate with common focus being to get onto arbitrary points on the strange attractor of Duffing's oscillator. Some of these attractors are reported in figures 5 to 9.

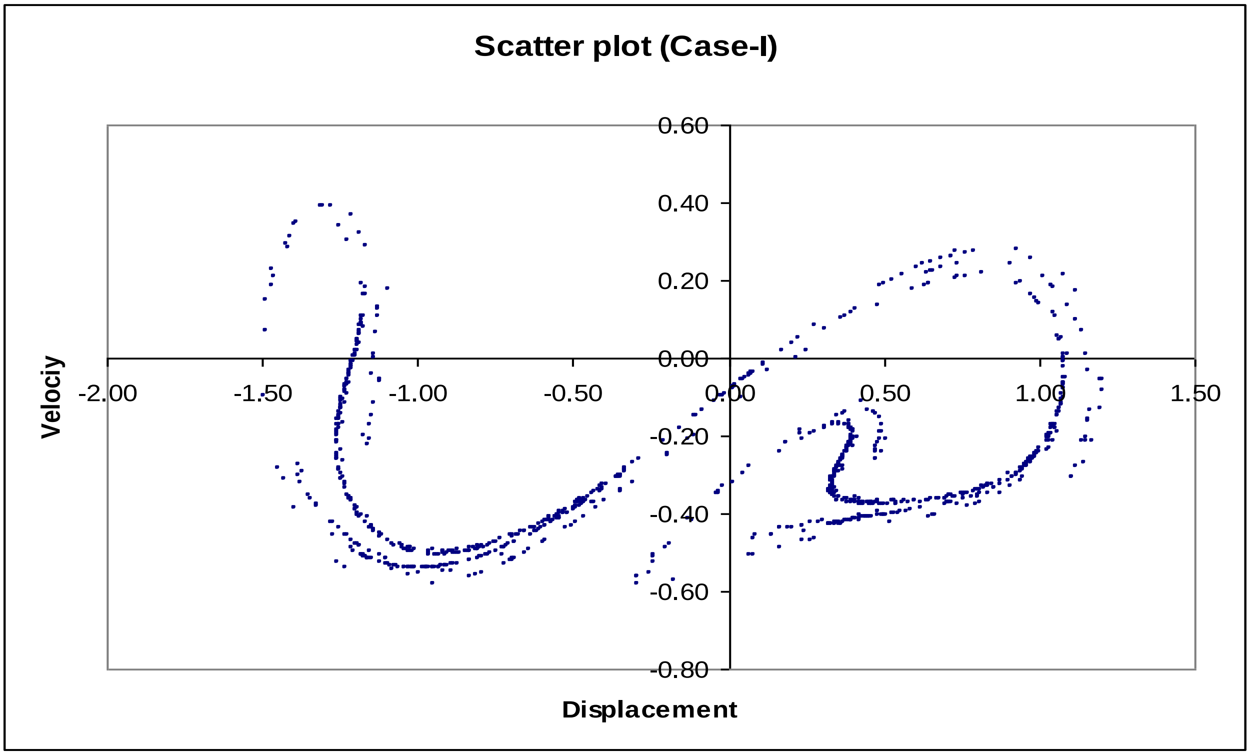

Figure 5: Steady scatter plot (Case-I) phase variables on phase plane computed at the end of 70th excitation period

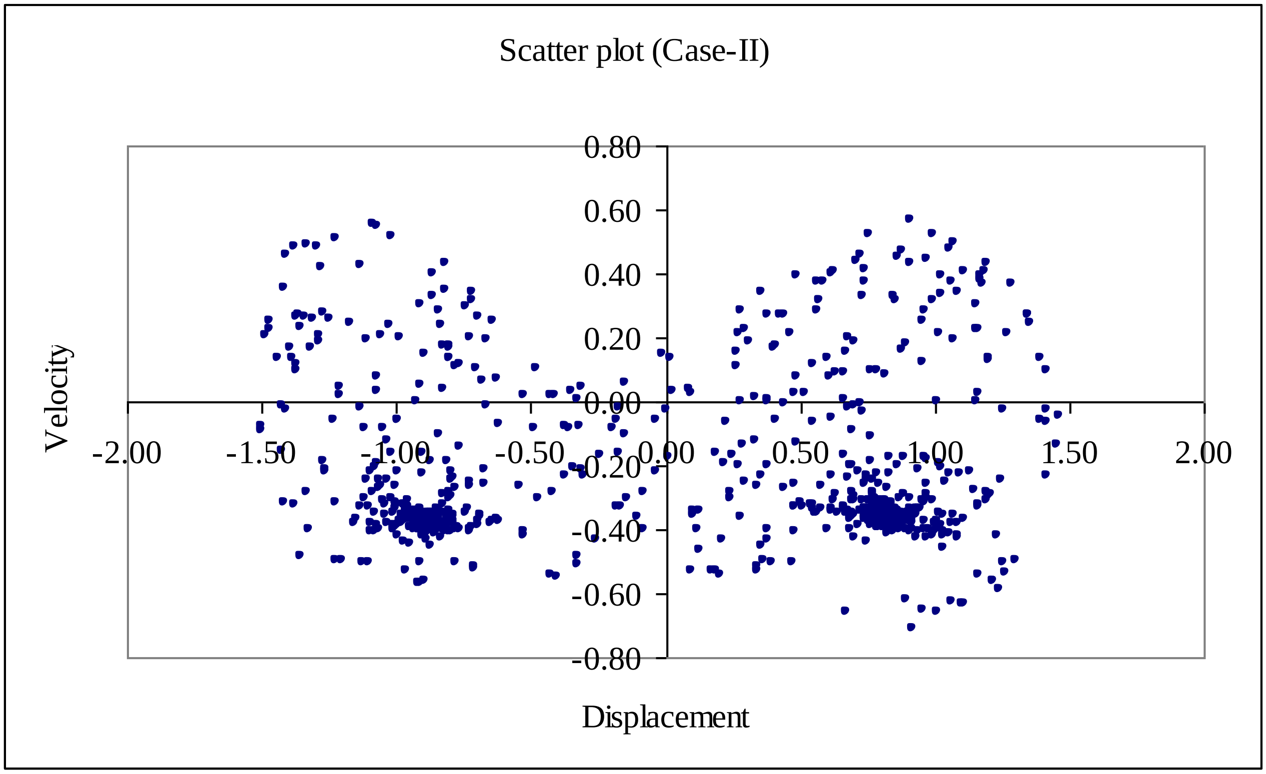

Figure 6: Steady scatter plot (Case-II) phase variables on phase plane computed at the end of 70th excitation period

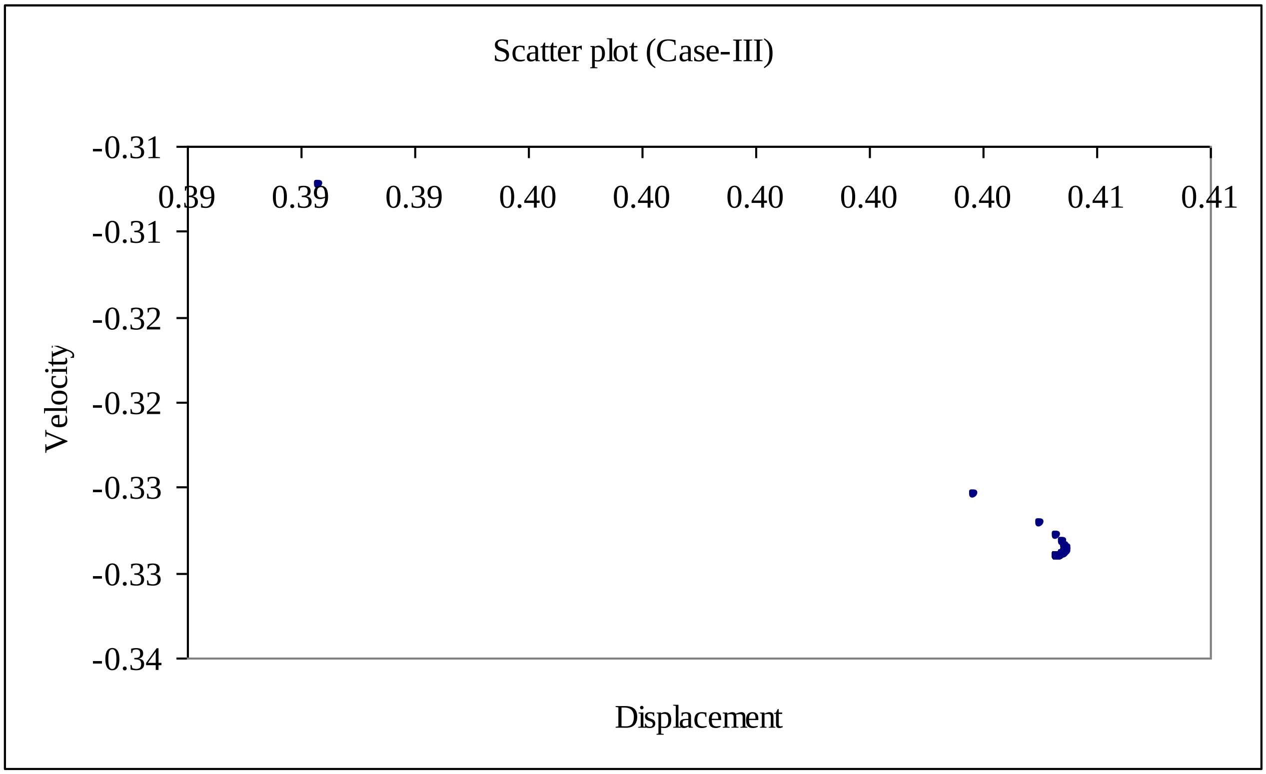

Figure 7: Steady scatter plot (Case-III) phase variables on phase plane computed at the end of 70th excitation period

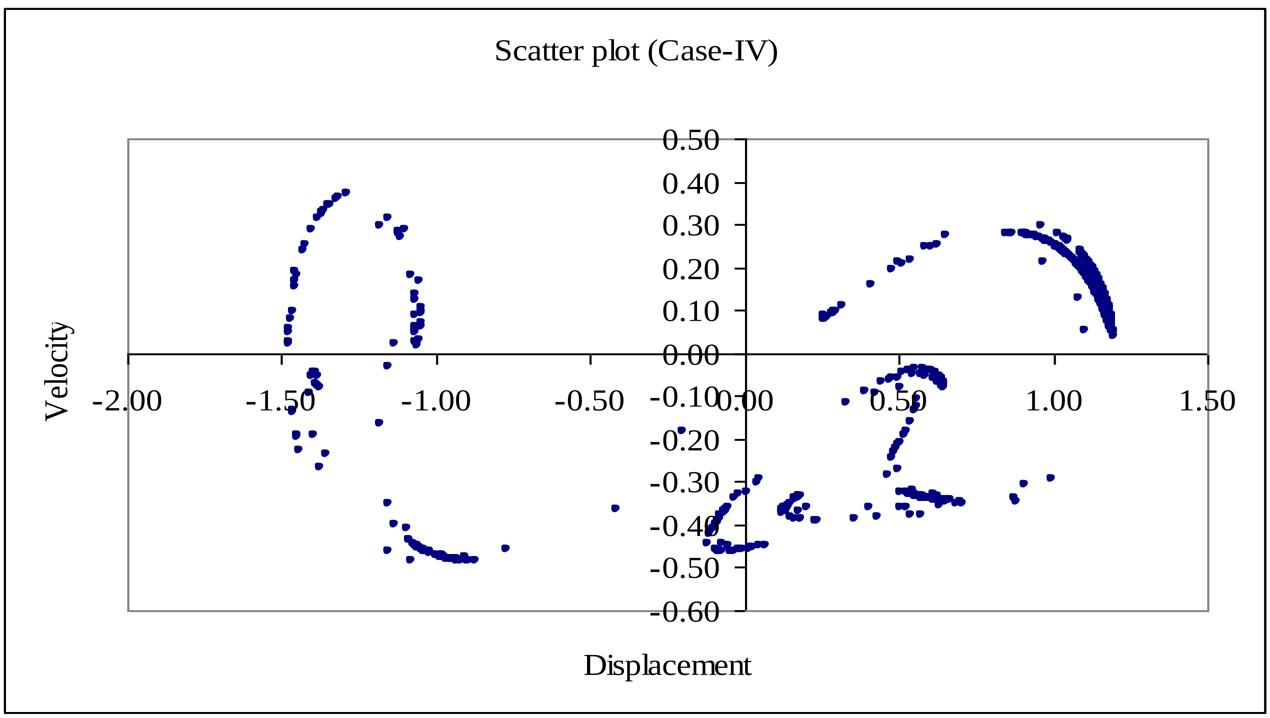

Figure 8: Steady scatter plot (Case-IV) phase variables on phase plane computed at the end of 70th excitation period

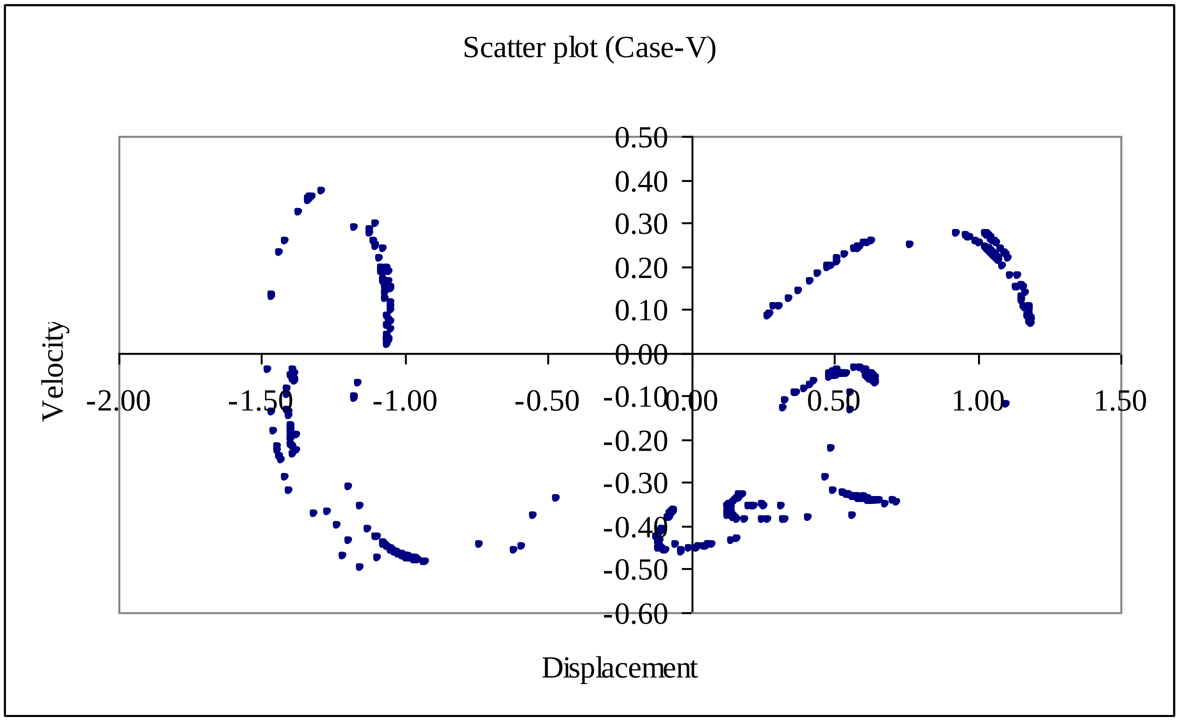

Figure 9: Steady scatter plot (Case-V) phase variables on phase plane computed at the end of 70th excitation period

The strange structured and bounded images of figures 5 and 6 are qualitatively acceptable within the accuracy limit of second-order Runge-Kutta methods as equivalent to the Poincare sections reported by Dowell (1988) and Salau and Ajide (2012 $_{(c)}$) and for respective corresponding driven parameters combination. Therefore it can be concluded that equivalent Poincare section can be obtained by simulation of equation (1) either from set of very close multiple initial conditions or by implementation of large number of varied version of Runge-Kutta methods. The standardised method of generating Poincare sections used by Dowell (1988) being the stroboscopic report of predicted phase variables at interval of one excitation length period performed repeatedly over infinite time length. Figure 7 match fairly the finite average dots number of period four response reported by Dowell (1988). However figures 8 and 9 deviated significantly from respective period two and one that Dowell (1988) predicted. The noticeable deviation can be accounted to lower instead of higher order Runge-Kutta methods used. The phase plot in figure 2 elucidates inaccuracy of adopted solution method and why figures 8 and 9 appeared disjointed unlike their figure 5 counterpart.

4. CONCLUSIONS

This study re-affirms complicated and varied nature of solutions to governing equation of Duffing's oscillator as change from one set of driven parameters combination to another are effected. Through use of scatter plot of the phase variables obtained from versions of Runge-Kutta methods emerge structured and bounded strange images on the phase plane that qualitatively replicate Poincare sections reference to literature. The observation was consistent especially for driven parameters combination that guarantees chaotic response of the Duffing's Oscillator. The utility of the procedures of this study can be in the preliminary verification of chaotic behaviour of dynamic systems subject to confirmation by additional acceptable dynamic tests.

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