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ABSTRACT

Let \( G^2 = (V, E(G^2)) \) be a square graph of a graph \( G \). A set \( D \) of \( G^2 \) is said to be connected dominating set of \( G^2 \), if every vertex not in \( D \) is adjacent to at least one vertex in \( D \) and the sub graph \( \langle D \rangle \) is connected. The minimum cardinality of a connected dominating set of \( G^2 \) is called the connected domination number of square graph \( G^2 \) and is denoted by \( \gamma_c(G^2) \). In this paper many bounds on \( \gamma_c(G^2) \) are found in terms of elements of \( G \) but not the elements of \( G^2 \). Also its relationship with other different domination parameters were obtained. Further we develop the relation between \( G \) and \( G^2 \) in terms of domination parameters.

Keywords: Graph, Square graph, Dominating set, Connected domination number

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1. INTRODUCTION

In this paper we follow the notations of [1]. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

In general, we use \( \langle X \rangle \) to denote the sub graph induced by the set of vertices \( X \) and \( N(v) \) and \( N[v] \) denote the open and closed neighborhoods of a vertex \( v \), respectively. Also the term \( \alpha_0(G)(\alpha_1(G)) \) is the minimum number of vertices (edges) in a vertex (edge) cover of \( G \).

The square of a graph \( G \), denoted by \( G^2 \), has the same vertices as in \( G \) and the vertices \( u \) and \( v \) are joined in \( G^2 \), if and only if they are joined in \( G \) by a path of length one or two. The concept of squares of graphs was introduced in [2].

Let \( G = (V, E) \) be a graph. In this paper we denote the number of vertices of \( G \) as \( p \) and \( q \) the number of edges. A set \( D \subseteq V \) is said to be a dominating set of \( G \), if every vertex in \( (V - D) \) is adjacent to some vertex in \( D \). The minimum cardinality of vertices in such a set is called the domination number of \( G \) and is denoted by \( \gamma(G) \).

A dominating set \( D^1 \) of \( G \) is said to be a connected dominating set, if the sub graph \( \langle D^1 \rangle \) induced by \( D \) is connected in \( G \). The minimum cardinality of vertices in such a set is called the connected domination number of \( G \) and is denoted by \( \gamma_c(G) \). Domination related parameters are now well studied in graph theory [3].

A dominating set \( D^1 \) of \( G \) is said to be an independent dominating set if no two vertices in \( D^1 \) are adjacent in \( G \). The minimum cardinality of an independent dominating set is called an independent domination number of \( G \) and is denoted by \( \gamma_i(G) \).

A dominating set \( D^1 \) of \( G \) is said to be total dominating set of \( G \), if \( N(D^1) = V \) or equivalently, if for every vertex \( v \in V \), there exists a vertex \( u \in D^1 \) such that \( u \) is adjacent to \( v \). The total domination number of \( G \), denoted by \( \gamma_t(G) \) is the minimum cardinality of a total dominating set of \( G \).

Further \( \beta_0(G) \) and \( \beta_1(G) \) represents vertex independence number and edge independence number.
A set $D$ of $G^2$ is said to be connected dominating set of $G^2$ if every vertex not in $D$ is adjacent to at least one vertex in $D$ and the set induced by $D$ is connected. The minimum cardinality of a connected dominating set of $G^2$ is called connected domination number of $G^2$. Much effort has been made by many authors to establish the relationship among distance domination parameters, for the known results see a recent survey by M.A.Henning [4].

In this paper we establish some relation of $\gamma_c(G^2)$ with $\gamma(G^2)$, $\alpha_0(G)$ and also the results of $\gamma_c(G^2)$ are expressed not in terms of the elements of $G^2$, but expressed in terms of the elements of $G$.

2. RESULTS

Initially we provide an upper bound for $\gamma_c(G^2)$ in terms of $\gamma_c(G)$.

**Theorem 2.1:** For any connected graph $G$, $\gamma_c(G^2) \leq \gamma_c(G)$. Equality holds for $\text{diam}(G) \leq 2$, except for $K_{p_1,p_2}$ with $p_1, p_2 \geq 2$.

**Proof:** For any connected graph $G$ with $\text{diam}(G) \geq 3$, let $P$ be a diametral path such that $P: v_1, v_2, \ldots, v_k$. Suppose $I = \{v_i, v_{i+1}, \ldots, v_k\}$, $i \leq k$ be a connected dominating set of $G$ such that $I \subseteq V(P)$, so that $|I| = \gamma_c(G)$. Now we consider $D = \{v_{i-1}, v_{i+1}, \ldots, v_{i+m}\}$. Hence $|D| = \gamma_c(G^2)$. Clearly, $\gamma_c(G^2) \leq \gamma_c(G)$.

Suppose $\text{diam}(G) \leq 2$ and if $G \cong K_{p_1,p_2}$, $p_1, p_2 \geq 2$ with $p_1 + p_2 = p = |V(G)|$. Then in this case, $|D| = |v_1| = |v_2|$, such that $p_1 \leq p_2$ or $p_2 \leq p_1$ respectively. Now in $G^2$, since $\text{diam}(G) \leq 2$, clearly, $|D| = 1|D'|$, and hence $\gamma_c(G^2) < \gamma_c(G)$.

Suppose $\text{diam}(G) \leq 2$ and if $G \cong K_{p_1,p_2}$. Then in this case, $|D| = 1|D'|$. Therefore $\gamma_c(G^2) = \gamma_c(G)$, gives the required result for equality.

Now we develop the relation of connected domination of $G^2$ with the spanning trees of $G$.

**Theorem 2.2:** For any nontrivial connected graph $G$, $\gamma_c(G^2) = \max \{\gamma_c(T^2) - 1\}$ where the maximum is taken over all spanning trees $T$ of $G$.

**Proof:** Let $G$ be a nontrivial connected graph and $T$ be a spanning tree of $G$. Then any connected dominating set of $T^2$ is also a connected dominating set of $G^2$. Hence $\gamma_c(G^2) \leq \gamma_c(T^2)$. Thus we have that $\gamma_c(G^2) \leq \max \{\gamma_c(T^2) - 1\}$, where maximum is taken over all the spanning trees $T$ of $G$.

For equality, we consider the reverse inequality of the above. If $G$ is a tree, then the Theorem holds trivially. So we assume that $G$ is connected graph containing cycles. Let $D$ be a minimum connected dominating set of $G^2$ and $C$ be a cycle in $G$. If we can prove that $D$ is also a connected dominating set of $G$ for $e \in E(C)$, then $\gamma_c(G-e^2) \leq D = \gamma_c(G^2)$. By applying this process a finite number of times, we have $\gamma_c(T^2) \leq \gamma_c(C^2)$, where the maximum is taken over all spanning trees of $T$ of $G$.

If $V(C) \subseteq V(D)$, then obviously $V(D) - e$ for any $e \in E(C)$ is also connected and the vertices in $V(C) - D$ are also within distance two to $D$.

If $V(C) \not\subseteq V(D)$, then we select an edge $xy$ in $C$ such that $\text{dist}(x,D) + \text{dist}(y,D) = \max \{\text{dist}(u,D) + \text{dist}(v,D)\}$ such that $uv \in E(C)$ in $G$. Now we will show that $D$ is connected dominating set of $G - \{xy\}^2$.

For any two adjacent vertices $u$ and $v$ in $G$ we have $|\text{dist}(u,D) - \text{dist}(v,D)| \leq 1$. Then if there exists a vertex $w$ in $V(C)$ of $G$ such that $\text{dist}(w,D) = \max \{\text{dist}(v,D)\}$. Now we say that $w = x$ or $w = y$. Without loss of generality suppose that $\text{dist}(x,D) = \max \{\text{dist}(v,D): v \in V(C)\}$. 


Let $z$ be another neighbor vertex of $x$ different from $y$ in $V(C)$. This gives immediately that $\text{dist}(z, D) \leq \text{dist}(y, D)$ in $G$. Thus we get the distance between a vertex in $V(G) - D$ and $D$ are not influenced by deleting the edge $xy$. This gives $\text{dist}(v, D)$ in $G - \{xy\}$, $xy \in E(C)$ is equal to $\text{dist}(v, D)$ such that $v \in V(G)$. Hence $D$ is also a connected dominating set of $G - e$ for some cycle edge $e$.

$\Box$

Now we give the following Proposition which is straightforward.

**Proposition 2.1:** For any $(p, q)$ graph $G$ with $p \geq 2$ vertices, if $\text{rad}(G) \leq 2$ then $\gamma_c(G^2) = 1$.

In the following Theorems we give lower bounds to $\gamma_c(G^2)$.

**Theorem 2.3:** For any nontrivial connected $(p, q)$-graph $G$ with maximum degree $\Delta$, then $\gamma_c(G^2) \leq \min\{1, p - (\Delta + 2)\}$.

**Proof:** By Theorem 2.2, it is sufficient to show that $\gamma_c(G^2) \leq \min\{1, p - (\Delta + 2)\}$ for any spanning tree $T$ with maximum degree $\Delta = \Delta(T)$.

If $\text{rad}(T) \leq 2$, then by Proposition 2.1, we get $\gamma_c(T^2) = 1$. Now we assume that $\text{rad}(T) > 3$. Let $P$ be any longest path in $T$ with end vertices $u$ and $v$. Then there exist two vertices $x$ and $y$ of $P$ such that $\text{dist}(x, u) = 2$ and $\text{dist}(y, v) = 2$ in $T$. Let $P'$ be a $xy$ subpath of $P$ and let $D' = V(P) - V(P')$. Let $D = V(T) - [D' \cup M(T)]$ where $M(T)$ is the set of all end vertices of $V(T)$. Hence $D$ must contain a connected dominating set of $T^2$. Since $u, v \in D' \cap M(T)$ and $M(T) \geq \Delta$, we have

\[
\gamma_c(T^2) \leq |V(T)| - |D' \cup M(T)| \leq |V(T)| - |D'| + |D' \cap M(T)| \leq p - 2^2 - \Delta + 2 \leq p - (\Delta + 2)
\]

as required.

$\Box$

**Theorem 2.4:** If $G$ is connected graph, then $\gamma_c(G^2) \leq 5\gamma(G^2) - 4$.

**Proof:** Let $G$ be a connected graph and let $D$ be a dominating set of $G^2$. Then the induced sub graph $\langle D \rangle$ has at most $|D|$ components. Since $D$ is a dominating set of $G^2$, we can connect two of these components to one component by adding at most four vertices to $D$. Hence we can construct a connected dominating set $D$ of $G^2$ such that $D \supseteq D$ in at most $|D| - 1$ steps by adding at most four times of $(|D| - 1)$ vertices to $D$. Consequently, $\gamma_c(G^2) \leq |D| \leq 4(|D| - 1) = 5|D| - 4$ and if we choose $D$ such that $|D| = \gamma(G^2)$.

In the following Theorem, we obtain the inequality relation between domination and connected domination of $G^2$.

**Theorem 2.5:** For any connected graph $G$, $\gamma(G^2) \leq \gamma_c(G^2)$. Equality holds for connected spanning sub graph $H$ of $G$ with $\text{diam}(H) \leq 4$.

**Proof:** For $p \leq 3$, the result is obvious. For $p \geq 4$, let $D = \{v_1, v_2, v_3, \ldots, v_m\} \subseteq V(G^2)$ be the set of vertices in $G^2$ such that $\text{diam}(u, v) \leq 4$, $u, v \in D$, which forms a minimal dominating set of $G^2$. Suppose $\text{diam}(u, v) = 4$, $u, v \in D$, then $D$ itself is a $\gamma$-set of $G^2$. Further, if $\text{diam}(u, v) > 4$, there exists at least one vertex $w \not\in D$, such that the sub graph $\langle D \cup \{w\} \rangle$ is connected. Clearly, $D \cup \{w\} = D$ forms a minimal $\gamma_c$-set of $G^2$. Therefore, it follows that $\gamma(G^2) \leq \gamma_c(G^2)$.

Suppose $\text{diam}(H) \leq 4$ where $H$ is spanning sub graph of $G$ and is connected. Then in this case, there exist a vertex $v \in V(G^2)$ which covers all the vertices of $G^2$. Clearly, it follows that $\gamma(G^2) = |\{v\}| = \gamma_c(G^2)$.
Theorem 2.6: For any nontrivial tree $T$ with $m$ end vertices, $\gamma_c(T^2) \geq \left\lfloor \frac{p-m}{3} \right\rfloor$.

Proof: Let $M = \{v_1, v_2, v_3, \ldots, v_k\}$ be the set of all end vertices in $T$ with $|M| = m$. Since $V(T) = V(T^2)$, without loss of generality in $T^2$ there exists a vertex set $D = V - M = \{v_1, v_2, v_3, \ldots, v_k\}$ which are at distance at most two and they covers all the vertices in $T^2$ such that the sub graph $\langle D \rangle$ is connected. Clearly, $D$ is a minimal connected dominating set of $T^2$. Further, if every tree with at least two end vertices and with $\text{diam}(T) \leq 2$, which generates
\[ \left\lfloor \frac{p-m}{2+2-1} \right\rfloor \leq |D| \] and hence\[ \frac{p-m}{3} \leq \gamma_c(T^2). \]

Theorem 2.7: For any connected $(p, q)$-graph $G$, $\gamma_c(G^2) \leq p - \alpha_q(G)$. Equality holds for $K_p$.

Proof: For $p = 2$, the result follows immediately. For $p \geq 3$, let $A = \{v_1, v_2, v_3, \ldots, v_k\}, \text{deg}(v_i) \geq 2, 1 \leq i \leq k$ be the minimal set of vertices which covers all the edges in $G$, such that $|A| = \alpha_q(G)$. Now without loss of generality in $G^2$, if $\text{deg}(v_i) \leq 2$ in $G^2$, then the result follows immediately. Further, if $\text{deg}(v_i) > 2$ in $G^2$, then there exists a vertex set $D = \{v_1, v_2, v_3, \ldots, v_m\}, \text{deg}(v_i) \geq 3, 1 \leq n \leq m$, which covers all the vertices of $G^2$ such that the sub graph $\langle D \rangle$ is connected. Clearly, $D$ is a $\gamma_c$-set of $G^2$. It follows that $|D| \leq p - |A|$ and hence $\gamma_c(G^2) \leq p - \alpha_q(G)$.

Suppose $G$ is isomorphic to $K_p$, then in this case, $|D| = 1$ and $|A| = p-1 = p - |D|$. Therefore, $\gamma_c(G^2) = p - \alpha_q(G)$.

Theorem 2.8: For any connected $(p, q)$-graph $G$, $\gamma(G) + \gamma_c(G^2) \leq p - 1$, for $p \geq 3$. Equality holds for $C_3, P_3, C_4$ and $P_4$.

Proof: For $p = 2$, clearly, $\gamma(G) + \gamma_c(G^2) \leq p - 1$. For $p \geq 3$, let $S = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\text{deg}(v_i) \geq 2, 1 \leq i \leq n$, be the minimal set such that $N[v_1] = V(G)$. Clearly $S$ is a minimal dominating set of $G$. Now without loss of generality in $G^2$, since $V(G) = V(G^2)$, there exists a set $D = \{v_1, v_2, v_3, \ldots, v_k\} \subseteq S$ which covers all the vertices in $G^2$. Suppose distance between two vertices $u, v \in D$ is at most two. Then the sub graph $\langle D \rangle$ itself is a connected and hence $D$ is a connected dominating set of $G^2$. Further if $\text{dist}(u, v) \geq 3$ where $u, v \in D$, and clearly, the sub graph $\langle D \rangle$ is disconnected. Then there exists at least one vertex $w \notin D$ such that $D \cup \{w\}$ forms a connected dominating set in $G^2$. Since distance between any two vertices in $G$ or $G^2$ is at least one, it follows that $|S| \cup |D \cup \{w\}| \leq p - 1$. Therefore $\gamma(G) + \gamma_c(G^2) \leq p - 1$.

For the equality, we have following cases.

Case 1: Suppose $G$ is isomorphic to $C_3$ or $P_3$. Then in this case, $|S| = |D| = 1$ and $|S| \cup |D| = p - 1$. Clearly, $\gamma(G) + \gamma_c(G^2) = p - 1$.

Case 2: Suppose $G$ is isomorphic to $C_n$ or $P_n$. Then in this case, $|S| = 2|D|$. Clearly, it follows that $\gamma(G) + \gamma_c(G^2) = p - 1$.

Corollary 2.1: For any connected $(p, q)$-graph $G$, $\gamma(G) + \gamma_c(G^2) \leq p - 1$, for $p \geq 3$. Equality holds for $C_3, P_3, C_4$ and $P_4$.

Theorem 2.9: For any connected $(p, q)$-graph $G$ with $p \geq 3$ vertices, $\gamma_c(G^2) \leq \left\lfloor \frac{p}{\Delta(G)-1} \right\rfloor$.

Proof: For $p = 2$, $\gamma_c(G^2) \leq \left\lfloor \frac{p}{\Delta(G)-1} \right\rfloor$. For $p \geq 3$, if $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of all non-end vertices in $G$. Then there exists at least one vertex $v \in V(G)$ such that $\text{deg}(v) = \Delta(G)$. Now without loss of generality in $G^2$, there exists a set
such that \( \text{deg}(v_i) \geq 2 \), \( 1 \leq i \leq k \) and the sub graph \( \langle D \rangle \) is connected in \( G^2 \).

It follows that \( D \) is \( \gamma_c \)-set of \( G^2 \). Since for any graph \( G \) there exists at least one vertex \( v_i \), \( \forall i \), \( 1 \leq i \leq k \) such that \( A \cap D = \{v_i\} \). Clearly, it follows that \( |D| \leq \left\lfloor \frac{p}{\Delta(G)-1} \right\rfloor \).

Therefore \( \gamma_c(G^2) \leq \left\lfloor \frac{p}{\Delta(G)-1} \right\rfloor \).

\[ \text{Corollary 2.2: For any connected \( (p,q) \)-graph \( G \) with \( p \geq 3 \) vertices, } \gamma_c(G^2) \leq \left\lfloor \frac{2q - p}{3} \right\rfloor. \]

The following theorem relates domination and connected domination numbers of \( G \) with \( \gamma_c(G^2) \).

**Theorem 2.10:** For any connected \( (p,q) \)-graph \( G \), \( \gamma_c(G) - \gamma(G) \leq p - \gamma_c(G^2) \).

**Proof:** Let \( S = \{v_1, v_2, v_3, \ldots, v_k\} \) be the minimal dominating set of \( G \). Suppose the sub graph \( \langle S \rangle \) is connected, then \( S \) is a connected dominating set of \( G \). Further if the sub graph \( \langle S \rangle \) is disconnected, then there exists another vertex set \( J = \{v_1, v_2, v_3, \ldots, v_i\} \) where \( J \subseteq V(G) - S \) whose vertices are at distance one to the vertices in \( S \) such that the sub graph \( \langle S \cup J \rangle \) is connected. Clearly, \( S \cup J \) is a connected dominating set of \( G \). Since \( V(G) = V(G^2) \), there exists a vertex set \( D = \{v_1, v_2, v_3, \ldots, v_k\} \subseteq S \cup J \), \( \text{deg}(v_i) \geq 2 \), \( 1 \leq i \leq k \), whose vertices are at distance at most one which covers all the vertices in \( G^2 \). Clearly, it follows that \(|S \cup J| - |S| \leq p - |D|\).

Therefore \( \gamma_c(G) - \gamma(G) \leq p - \gamma_c(G^2) \).

**Theorem 2.11:** For any connected \( (p,q) \)-graph \( G \), \( \frac{\gamma_c(G^2)}{2} \leq p - \text{diam}(G) \), except for path \( P_p \) with \( p > 7 \). Equality holds for path \( P_p \) with \( p \leq 7 \).

**Proof:** Suppose \( G \cong P_p \) with \( p > 7 \), then \( \frac{\gamma_c(G^2)}{2} \leq p - \text{diam}(G) \). Let \( V(G) \) be the set of vertices in \( G \) such that there exists a diametral path between two vertices \( u, v \in G \), where \( \text{dist}(u, v) \) forms a \( \text{diam}(G) \). Let \( D = \{v_1, v_2, v_3, \ldots, v_n\} \), \( \text{deg}(v_i) \geq 2 \), \( 1 \leq i \leq n \) in \( G^2 \), which are at distance at least two. These vertices covers all the vertices in \( G^2 \). Suppose the sub graph \( \langle D \rangle \) connected, then \( D \) itself is a connected dominating set of \( G^2 \). Further, if the sub graph \( \langle D \rangle \) is disconnected, then there exists at least one vertex \( w \notin D \) which is at distance at most two to the vertices in \( G^2 \). Clearly, the sub graph \( \langle D \cup \{w\} \rangle \) is connected and the set \( D \cup \{w\} \) is a minimal \( \gamma_c \)-set of \( G^2 \). Since any two vertices \( u, v \in G \) forms a diametral path of at least one in \( G \), it follows that \( \left\lfloor \frac{D \cup \{w\}}{2} \right\rfloor \leq p - \text{diam}(G) \).

Hence, \( \frac{\gamma_c(G^2)}{2} \leq p - \text{diam}(G) \).

Suppose \( G \) is isomorphic, path \( P_p \) with \( p \leq 7 \). Then in this case, \( |D \cup \{w\}| = p - 2 \) and \( \text{diam}(G) = p - 1 \). Clearly, it follows that,

\[ \frac{\gamma_c(G^2)}{2} = 1 = p - \text{diam}(G). \]

**Corollary 2.3:** For any nontrivial tree \( T \) with \( \text{diam}(T) \leq 4 \), then \( \gamma_c(T^2) = 1 \).

Further we obtain the connected domination number of squares of some standard graphs as:

\[ \gamma_c(K_{p,q}^2) = \gamma_c(W_5^2) = \gamma_c(K_{1,p}^2) = \gamma_c(K_{2,p}^2) = 1. \]

The following theorem relates connected domination and total domination numbers of \( G^2 \).

**Theorem 2.12:** For any connected \( (p,q) \)-graph \( G \) with \( p \geq 3 \), \( \gamma_c(G^2) + \gamma_t(G) \leq p \). Equality holds for path \( P_3, P_6, C_3, C_6 \).
Proof: For \( p = 2 \), clearly, \( \gamma_c(G^2) + \gamma_c(G) \leq p \).

For \( p \geq 3 \), if \( S = \{v_1, v_2, v_3, \ldots, v_n\} \) be the minimal set of vertices with \( \deg(v_i) \geq 2, 1 \leq i \leq n \) and \( N[v_i] = V(G) \). And if the sub graph \( \langle S \rangle \) is not disconnected then

\( S \) is itself a minimal total dominating set of \( G \). Otherwise, \( S \cup H \), where \( H \subseteq V - S \), covers all the vertices in \( G \) and the sub graph \( \langle S \cup H \rangle \) doesn’t contain any isolated vertex. Clearly, the set \( S \cup H \) forms a minimal total dominating set of \( G \). Since \( V(G) = V(G^2) \), there exists a vertex set \( D = \{v_1, v_2, v_3, \ldots, v_k\} \subseteq S \) in \( G^2 \) which covers all the vertices in \( G^2 \) and the sub graph \( \langle D \rangle \) is connected.

Hence \( D \) forms a minimal \( \gamma_c \)-set of \( G^2 \). It follows that \( |S \cup H| \cup |D| \leq p \) and hence \( \gamma_c(G) + \gamma_c(G^2) \leq p \).

For equality, we have the following cases.

Case 1: Suppose \( G \) is isomorphic to \( C_3 \) or \( P_3 \). Then in this case \( |S \cup H| = 2 \) \( |D| \) and \( |D| = 1 \). Clearly, it follows that \( \gamma_c(G) + \gamma_c(G^2) = p \).

Case 2: Suppose \( G \) is isomorphic to \( C_6 \) or \( P_6 \). Then in this case \( |D| = 2 \) and \( |S \cup H| = 2 |D| \). Clearly, it follows that \( \gamma_c(G) + \gamma_c(G^2) = p \).

The following result gives the relation of connected domination of \( G^2 \) in terms of vertices and maximum number of independent vertex of \( G \).

Theorem 2.13: For any connected \( (p, q) \)-graph \( G \), \( \gamma_c(G^2) \leq p - \beta_0(G) \).

Proof: Suppose \( F = \{v_1, v_2, v_3, \ldots, v_n\} \) be the set of all end vertices in \( G \) then \( F \cup B \) where \( B \) is a proper subset of \( V(G) - F \), which are not adjacent to the vertices of \( F \) forms a minimal independent set of vertices such that \( |F \cup B| = \beta_0(G) \). Now in \( G^2 \), since the distance of \((u, v) \geq 2\) for all \( u, v \in V(G^2) \) and induced sub graph \( B \) is connected and vertices of \( B \) covers all the vertices in \( G^2 \). Clearly, \( B \) self is a \( \gamma_c \)-set of \( G^2 \). Otherwise there exists at least one vertex \( w \notin B \) in \( G^2 \), such that the sub graph \( \langle B \cup \{w\} \rangle \) forms a connected sub graph in \( G^2 \). Therefore it follows that \( |B \cup \{w\}| \leq p - |F \cup B| \) and hence \( \gamma_c(G^2) \leq p - \beta_0(G) \).

Theorem 2.14: Let \( G \) be a connected graph and \( H \) be any connected spanning sub graph of \( G \). Then every connected dominating set of \( H^2 \) is also a connected dominating set of \( G^2 \) and hence \( \gamma_c(G^2) \leq \gamma_c(H^2) \).

Proof: Suppose \( H \) is totally disconnected or disconnected with at least one component as an isolated vertex. Then we consider \( V(G) = \{v_1, v_2, v_3, \ldots, v_n\} \) and \( S = \{v_i\} \), \( 1 \leq i \leq n \) such that \( S \subseteq V(G) \) in such a way that every vertex of \( V - S \) are at a distance of at most two with respect to the corresponding vertices of \( S \) in \( G \) which gives a connected minimal connected dominating set in \( G^2 \). Let \( v_j \) be an edge of \( G \) such that \( i < j \) and \( \forall i, j = 1, 2, \ldots, (n-1) \). Suppose \( H \) is a minimal connected spanning sub graph of \( G \) and \( E(H) = E(G) - v_i v_j \). Since \( V(G^2) = V(H^2) \) then \( S \) is also a minimal connected dominating set of both \( G^2 \) and \( H^2 \) which gives the equality \( \gamma_c(G^2) = \gamma_c(H^2) \). If \( H \) is totally disconnected then \( V(G^2) = V(H^2) \) such that \( E(H^2) = \phi \) and \( \gamma_c(H^2) = V(G^2) = n \), where \( n \) is the number of vertices in \( G \). Now one can easily verify that \( \gamma_c(G^2) < \gamma_c(H^2) \). Otherwise \( \gamma_c(G^2) \leq \gamma_c(H^2) \).

Finally we provide Nordhaus – Gaddum type result.

Theorem 2.15: For any connected \( (p, q) \)-graph \( G \),

(i) If both \( G^2 \) and \( G^2 \) are connected, then
(a) \( \gamma_c(G^2) + \gamma_c(G^2) \) \leq p + 1.
(b) \( \gamma_c(G^2) \cdot \gamma_c(G^2) \) \leq p.

(ii) If both \( G^2 \) and \( G^2 \) are connected, then
(a) \( \gamma_c(G^2) + \gamma_c(G^2) \leq p + 1. 
(b) \gamma_c(G^2) \cdot \gamma_c(G^2) \leq p.
REFERENCES


